NOTES

Frontispiece: The Tower of Mathematics is the Tower of Babel inverted: its voices grow more coherent as it rises. This image of it is based on Pieter Brueghel's painting "Little Tower of Babel" (1554).

AN INVITATION

1: [Passage from Blake] This is from Proposition V in the Second Series of Blake's *There Is No Natural Religion*, which reads in full: "If the many become the same as the few when possess'd, More! More! is the cry of a mistaken soul; less than All cannot satisfy Man."

1: [Baudelaire] "*Berçant notre infini sur le fini des mers*": *Le Voyage*, I. 8.

1: [Mahāvīrā passage] Quoted in Katz, p. 219.

2: [Hawking] "Someone once told me that each equation I included in my book would halve the sales." Apparently in the preface to the first edition of his *Brief History of Time*.

ONE: TIME AND THE MIND

3: [Number knowledge among animals] See Paul Recer, "Monkeys Show Number Sense," in Abcnews.com Science, Oct. 23 1998 (http://abcnews.go.com/sections/science/DailyNews/monkeys981022.html).

4: [Wunnery tooery] From James T. R. Ritchie's *Golden City: Scottish Children's Street Games and Songs* (Edinburgh: Mercat Press, 1999), p. 39. There are numerous counting-out rhymes here, almost all demonstrating at the same time a mastery of counting and the mystery of our numerals. See too Peter and Iona Opie's wonderful *Language and Lore of Schoolchildren* (Oxford, 1959).

4: [Burla, ùù] James R. Hurford, *Language and Number* (Blackwell, 1987), pp. 51, 54.

5: [Passage from Stevenson] From his essay "Pulvis et Umbra" in the collection *Across the Plains.*

5: [Housman] *Last Poems,* XXXV.

6: [Oksapmin counting] Hurford (op. cit.), p. 81. See too G. B. Saxe, "Culture and the Development of Numerical Cognition: Studies Among the Oksapmin of Papua New Guinea," in *Children's Logical and Mathematical Cognition,* B. Brainerd (ed.; Springer 1982), pp. 157–76.

7: [Figures in Proust appearing first in asides] So Charlus is first seen by the narrator (in *Combray*) a little way off: "...a gentleman dressed in a suit of linen 'ducks', whom I didn't know either, stared at me with eyes which seemed to be starting from his head." (Pleiade edition, i, p.141). *Swann's Way*, translated by C. K. Scott Moncrieff (Modern Library, 1928), p. 202.

9: [Light precedes use] From Bacon's preface to his *The Great Instauration:* "And there is another thing to be remembered—namely, that all industry in experimenting has begun with proposing to itself certain definite works to be accomplished, and has pursued them with premature and unseasonable eagerness; it has sought, I say, experiments of fruit, not experiments of light, not imitating the divine procedure, which in its first day's work created light only and assigned to it one entire day..."(pp. 11–12 in the Library of Liberal Arts edition of *The New Organon and Related Writings,* Fulton H. Anderson [ed., Bobbs-Merrill, 1979]).

9: [Nameless mathematician] Theon of Smyrna, who reported this insight, may have been its discoverer. See Heath, i, p. 83.

9: [Heraclitus] Fragment 54: ἁρμονίη ἀφανὴς φανερῆς κρείττων.

11: [Twitchiness about asymmetries] Gerald Holton, on pp. 363–66 of his *Thematic Origins of Scientific Thought: Kepler to Einstein* (Harvard University Press, 1973), points out that it was precisely the discomfort Einstein felt at the asymmetry between the calculation of current when the conductor moves with respect to a stationary magnet, and when the conductor is held still and the magnet moves, that led him to the key notion of relativizing, and later to his General Theory of Relativity. He quotes from a 1919 ms. of Einstein's: "...The thought that one is dealing here with two fundamentally different cases was for me unbearable. The difference between these two cases could not be a real difference but rather, in my conviction, only a difference in the choice of reference points... A kind of objective reality could only be granted to *the electric and magnetic fields together*..." Holton remarks that Einstein's "desire to remove an unnecessary asymmetry was not frivolous or accidental, but deep and important," and connects his aesthetic need to simplify to the man's own legendary simplicity in life.

12: [Yesterday upon the stair] This verse is almost not there itself. It appears (under the rubric "Rootless Rhymes") in Roger Lancelyn Green's *A Century of Humorous Verse* (London: Dent and Sons, 1959) in this form:

> As I was coming down the stair
> I met a man who wasn't there.
> He wasn't there again to-day:
> I *wish* that man would go away!

In fact, however (if "fact" is the apposite word), the verse isn't anonymous and its sense is reversed. It is by Hughes Mearns (1875–1965), is called "The Psychoed," and reads:

> As I was walking up the stair
> I met a man who wasn't there.
> He wasn't there again today—
> I wish, I wish he'd go away.

The psychotic element referred to in the title is, of course, the wish to avoid this man, since it is widely believed unlucky to meet someone on the stairs (see the article "Stairs" in Opie and Tatem's *A Dictionary of Superstitions*); hence Mr. Mearns should have thanked his (missing) stars.

13: [Nichomachus's imagination] The "tongueless animals with but a single eye" were numbers, like 15, the sum of whose proper divisors was less than the number itself; the second sort—like 12—had proper divisors that summed to more than the number. For more on Nichomachus, see Heath, i, pp. 97–112; Kline pp. 135–38; and the St. Andrews web-site.

13: [Kronecker] *“Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.”* From a letter of Kronecker’s, quoted in Meschkowski, p. 136, footnote 187.

14: [Ox for loincloths, chicken for cowrie shells] Georges Ifrah, *From One to Zero: A Universal History of Numbers* (Penguin Books, 1987), p. 100.

16: [Pythagorean secrets, universal harmony] Kirk and Raven, p. 329: fragment 430 (Aristotle, *Meta*. A5 985b23). For a fuller treatment, see Heath, i, p. 86, and Christopher Scriba, *Mathematics and Music* (Normat, 1990), who thinks that the germ of the idea of these musical divisions was Babylonian.

16: [Greek merchant fractions] Kline, p. 32, Kaplan, *The Nothing That Is* (Oxford University Press, 2000), Chapter 2.

16: [The eye of Horus] According to Ifrah (op. cit., p. 168), in Egyptian mythology Seth ripped out the eye of Horus and tore it into six pieces, each of which stood for one of the special fractions used in Egyptian measures. These pieces, and the numerical values they stood for, were: the eyebrow (1/8); the pupil (1/4); the white of the eye toward the nose (1/2); the white of the eye toward the ear (1/16); the curling diagonal below the eye (1/32); and the vertical line below the eye (1/64). A doubly visual mnemonic?

16: [Great fleas...] These lines appear in Augustus de Morgan’s *A Budget of Paradoxes* (Dover reprint, 1954, of the second (1915) edition, Vol. 2, p. 191) with this note of the author's: "As Swift gave it in his *Poetry, A Rhapsody,* it is as follows:

So, naturalists observe, a flea
has smaller fleas that on him prey;
And these in turn have smaller still to bite 'em,
And so proceed *ad infinitum.*

The art of acknowledging, with improvements, is among those that are in decline.

18: [Pythagorean ratios and tetractys] Kirk and Raven, pp. 233–34, frag. 229 (Sextus Empiricus).

18: [Pythagorean significance of the first four numbers] Not only do the ratios of these numbers provide the fundamental musical intervals, but the individual numbers are fraught with significance. So 1 is the point, 2 the line, 3 the plane and 4 the solid (Heath, i, 76, citing Speusippus). And again, 1 is both even and odd (for being the principle of all number, it cannot be exclusively odd or even), 2 the principle of even, 3 of odd and also the first triangular number, 4 the first square, etc. (see Heath, i, pp. 67–77).

18: [Font and root of ever-flowing nature] Kirk and Raven, p. 233, note 1.

18: [Hippasus] Kline, p. 32. See Heath, i, pp. 154–55, 168 in general; and i, p. 154 on irrationality of $\sqrt{2}$; i, p. 155 on its irrationality first discovered by

Pythagoreans; i, p. 60 for story of shipwreck of Hippasus and of first discoverer of irrationality of √2. For more, see below, note to p. 189.

19: [√2 clumsy as a symbol] For the remarkable variety of attempts to make this symbol less clumsy—resulting only in stragglers limping across the snowy page—see Cajori, i, §316–38 and Smith, ii, pp. 407–10.

21: [√3 irrational] To show √2 wasn't rational we thought of the naturals as coming in two flavors: even and odd—or you could say, as leaving remainders of 0 or 1 when divided by 2. To show √3 irrational we need correspondingly to classify the naturals into three sorts: those leaving remainders of 0, 1 or 2 on division by 3. Then, arguing as before, if √3 = a/b (in lowest terms), then $3b^2 = a^2$ so a^2 is divisible by 3: a remainder of 0, that is, is left on dividing it by 3. We then need to show that a is therefore divisible by 3 also, and we do this by showing that if it weren't—if a was of the form 3n + 1 or 3n + 2—then so would a^2 have been. The proof then follows the now familiar format, reaching the contradiction that a and b both have 3 as a common factor (contradicting a/b having already been in lowest terms). To prove √5 irrational we argue similarly but now think of the five possible remainders on division by 5; and so on for √p, where p is prime.

22: [Ever narrowing cracks] Such an infinity as appealing rather than appalling was seen by the twelfth-century mathematician Al-Samaw'al, who invented an algorithm for better and better approximating various roots. "We are able by this method," he wrote, "to obtain answers infinite in number, each of which is more precise and closer to the truth than that which precedes it." Katz, p. 228.

24: [von Neumann] See David Wells, *The Penguin Book of Curious and Interesting Mathematics* (Penguin, 1997), p. 259.

24: [Bringing up the ruler] Compare Wittgenstein, *Tractatus Logico-Philosophicus* 2.1511: “That is how a picture is attached to reality; it reaches right out to it.” And 2.1512: “It is laid against reality like a measure.” (*“Es ist wie ein Masstab an die Wirklichkeit angelegt.”*)

25: [‘One’ seems there in the mind and its world] Many have argued that zero isn’t a number (see Kaplan, *The Nothing That Is*, passim) and some even denied this status to one. Thales seems to have thought that since number is defined as a collection of units, the unit itself could not be a number (Heath, i, p. 69). Aristotle (*Meta*. N. 1 1088a6) said that since one is the measure of number it can’t be a number itself; and this view is implied in Euclid (VII, defs. 1, 2) and may even have been held by the Pythagoreans (Heath, i, p. 69). In the third century B. C. Chrysippus made a brave attempt to make One, so understood, nevertheless a number, by calling it a ‘multitude one’: πλῆθος ἕν (Heath, i, p. 69). This is pre-reminiscent of Fregean attempts to define a number n as the set of all collections of n objects.

26: [Alcibiades and the carter] Plutarch’s *Lives*: Alcibiades, II.

26: [The power of the saved remnant] The origin of the tradition of a saved remnant is perhaps Rom 9:27: “Esaias also crieth concerning Israel, Though the number of the children of Israel be as the sand of the sea, a remnant shall be saved.” Compare Isaiah 6:13. Stories of such a saved remnant occasionally become confused or conflated with those of the Lamed Wuvniks, the thirty-six (because of the numerical values, 30 for lamed, 6 for vuv) righteous pillars of the universe, unknown to themselves or to others, often poor, it is said, and deranged. Perhaps they are paralleled by the Kutb in Islam. They are certainly well savored by Borges in his *Book of Imaginary Beings.* For more, see *Tractate Sanhedrin,* 97b. The immense power of reasoning in mathematics via the saved remnants of residues was invented by Gauss, and is known as modular arithmetic.

27: [Imaginaries as sophistic] Bombelli spoke of them so before understanding how to manipulate them. Kline, p. 253.

27: [Descartes coining the name ‘imaginary’] Kline, p. 253.

27: [Newton calling imaginary numbers ‘impossible’] Kline, p. 254.

27: [Leibniz calling imaginaries ‘amphibians’] Kline, p. 254.

27: [Passage from Girard] Kline, p. 253.

TWO: HOW DO WE HOLD THESE TRUTHS?

29: [Proofs of the Pythagorean Theorem] Both the Indian and Chinese proofs are in Howard Eves, *Great Moments in Mathematics (Before 1650)* (Mathematical Association of America, 1980), pp. 27–32.

30: [Ligature way of summing] You hit a bit of a snag if you try this when the last number is odd:

$$1 + 2 + 3 + 4 + 5 + 6 + 7$$

1 + 7 = 2 + 6 = 3 + 5 = 8 (that is, 7 + 1, or n + 1), with a solitary 4 left in the middle. So 4, which is 8/2, plus 3 pairs of 8. That is

$$\frac{(n+1)}{2} \quad + \quad (n+1) \quad \bullet \quad \frac{(n-1)}{2}$$

$$\frac{8}{2} \quad + \quad 8 \quad \bullet \quad \frac{6}{2}$$

But this simplifies to $\frac{(n+1)}{2} \bullet (1 + n - 1) = n \frac{(n+1)}{2}$, as before.

31: [Mind of ten-year-old Gauss] Bell, p. 221.

31: [Eudoxus] Kline, pp. 48, 50.

31: [Axioms] ‘Axiom’ is from the Greek ἀξίωμα, “something thought worthy or fit,” built on a lost noun ak-tis: weight, from ἄγειν = weigh or pull, and ultimately from the Indo-European ‘ag’.

31: [Aristotle hedging bets] Kline, p. 52.

31: [Stoics] The Greek καταληπτικὴ φαντασία has been variously translated as “recognizable presentation”, “cognative presentation”, “apprehension by the imagination”, etc. The core of the Stoic idea (as in Zeno, Chrysippus and Sextus Empiricus) is that the external object causes this impression, which is imaged and stored in the perceiver. See *Problems in Stoicism,* A. A. Long (ed.; The Athlone Press, 1971), pp. 9–21 and J. M. Rist, *Stoic Philosophy* (Cambridge, 1969), pp. 133–51.

31: [Clenched fist] Long (op.cit.), p. 11. Note that in his Inaugural Presidential Address to the Mathematical and Physical Section of the British Association, in August of 1869, J. J. Sylvester said: “Mathematical analysis is constantly invoking the aid of new principles springing from continually renewed introspection of the inner world of thought (analysis may be conceived to stand to the outer physical world as the hollow palm of one hand to the closed fist which it grips of the other)...”

32: [Inner or natural light] Descartes, pp. 295–302.

32: [Descartes’s dream] Kline, p. 306.

32: [Descartes’s life] Kline, p. 304.

32: [What Descartes said in the *Règles*] This is a précis of Descartes’s Rules 3, 7, and 9. How to translate the word “*intuere*” is the subject of much debate. J.-L. Marion (René Descartes (ed. J.-L. Marion), *Règles Utiles et Claires pour la Direction de l’Ésprit en la Recherche de la Verité* (Nijhoff, 1977), pp. 295ff, insists on avoiding “intuit” because of its non-Cartesian associations, using instead “*regarder*” in his French translation from Descartes’s Latin. Marion points out that Descartes, following St. Thomas, deliberately makes a passivity into an activity: “*Il transforme consciemment ainsi une passivité en une activité pour rester en accord avec le principe augustinien de ‘absence d’action des sens sur l‘esprit.’*”

32: [Portrait of Descartes] Based on the Louvre portrait.

32: [Gergonne] Quoted in Novy, p. 185, citing *Annales de mathematiques pur et appliqués,* Tome I (1810–11) pp. 815–16: "...*dont il suffit de connaître l'énoncer pour en apercevoir la verité*."

32: [Rimbaud] The passage is from the last section, *"Adieu": "…nous sommes engagés à la découverte de la clarté divine... Tous les souvenirs immondes s'effacent... il me sera loisible de* <u>*posséder*</u> *la verité dans une âme et un corps."*

32: [Locke and self-evident truth] Locke's 1690 is the first use quoted in the NED.

33: [Kant on the peculiarity of mathematics] *Prolegomena,* 1783.

33: [Ohm] Novy, p. 87.

34: [Peacock's Principle] Kline, p. 773; Novy, pp. 190–96. Certainly this principle would have grown in the congenial company of Charles Babbage, with whom (among others) Peacock formed the Analytical Society in Cambridge, England in 1812 (Katz, p. 678). Certainly Babbage would have endorsed the idea of feeding any sort of number into a machine and then simply cranking a handle, since he himself had invented "the analytical engine" (Kline, p. 259), an early calculating machine run whirr whirr all by steam.

34: [Portrait of Peacock] Based on portrait on St. Andrews website.

34: [Schumann's diary entry] The original, from Robert Schumann's *Tagebücher* (entry of June 8, 1831, I, p. 339) reads: "*Mir ist's manchmal. als wolle sich* *mein objectiver Mensch von subjectiven ganz trennen oder als ständ' ich zwischen meiner Erscheinung u. meinen Syn, zwischen Gestalt und Schatten*."

34: [Pascal] *Pensées,* p. 340.

35: [Folded in a tin box] J. Franklin Jameson, *An Introduction to the Study of the Constitutional and Political History of the United States* (Baltimore, 1886), quoted in Michael Kammen, *A Machine That Would Run of Itself* (Vintage, 1987), p. 127. Kammen takes his title, he says (p. 125), from James Russell Lowell, 1888.

35: [Distributive and Commutative Laws] These laws were first given their names in 1814 by François-Joseph Servois (Kline, p. 774).

35: [Precious only endless world] From Robert Graves's poem "Warning to Children".

35: [Russia, Hungary and Germany] Kline, pp. 870 ff. The Frenchmen included Jean-Victor Poncelet, whose projective geometry we will explore in Chapter Eight; Nicolai Ivanovitch Lobachevsky in Russia, János Bolyai in Hungary, and Bernhard Riemann in Germany came up with other approaches to non-Euclidean geometry as—inevitably—did Gauss.

35: [One attempt followed another] See Kline, pp. 862–67 for details.

35: [Scandal] So d'Alembert in 1759, who "called the problem of the parallel axiom 'the scandal of the Elements of Geometry'." (Kline, p. 867).

35: [*Un pur...*] Cited by Richard Cobb in one of his New College lectures, as noted by Michael Kaplan. Possibly said in reference to Robespierre and his fate. In any case, the aphorism is based (as Jon Tannenhauser astutely points out) on lines from the end of the first Canto in Boileau's *Art Poetique:* "*Un sot trouve toujours un plus sot qui l'admire."*

35: [An existence haunted by existence] Reid, p. 154.

36: [Hungarian anecdote] Paraphrased from Reid, pp.154–55.

37: [How '*Körper*' became 'field'] See http://members.aol.com/jeff570/f.html. From the website "Earliest Known Uses of Some of the Words in Mathematics" (http://members.aol.com/jeff570/f.html): "Julio González Cabillón believes that Eliakim Hastings Moore (1862–1932) was the first person to use the English word *field* in its modern sense..."

37: [Mathematicians evolving from sensation toward abstraction] Sylvester (op. cit., p. 105, footnote) speaks of "...sensation, perception, reflection, abstraction as the successive stages or phases of protoplasm on its way to being made perfect in Mathematical Man... We should then have four terms... the Vegetable, Animal, Rational, and Supersensual modes of existence."

38: [Tablets of the law delivered by Weber] Weber's version of the field axioms followed on earlier approaches, from different standpoints, by Galois, Abel, Dedekind, Kronecker, and others. They appeared almost in this form in Dedekind's Supplement X to his edition of Dirichlet's *Lectures on Number Theory*, in 1871 (see Katz, pp. 676–77). In his axioms for the real numbers of

1899 Hilbert incorporated these axioms, protesting against Peano's "genetic approach" of building the axioms up from those of the naturals. See Kline, pp. 990–92.

39: [Hamilton and pinning down irrationals] Kline, p. 983.

39: [Dedekind's diary] St. Andrews website: Dedekind.

39: [Dedekind cut] Edwards, "Dedekind's Invention of Ideals" (in Phillips, pp. 15–17). John Stillwell points out (personal communication) that Eudoxus's treatment of rationals, in fourth century B. C. Greece, lies in the background of Dedekind's cut notion, which Dedekind himself acknowledges in the preface to the first edition of his *Was sind und was sollen die Zahlen?* (trans. W. W. Beman): "...if... one regards the irrational number as the ratio of two measurable quantities, then is this manner of determining it already set forth in the clearest possible way in the celebrated definition which Euclid gives of the equality of two ratios (*Elements*, V, 5)." See Stillwell, p. 62.

39: [Portrait of Dedekind] Based on photograph of Dedekind in Walter Purkert and Hans Joachim Ilgauds, *Georg Cantor, 1845–1918* (Leipzig: B. G. Teubner, 1895), p. 52.

40: [Boston detective] Paraphrased from BBC television program "Boston Law," February 7, 2001.

41: [Cubic curve] We are indebted to Barry Mazur for calling our attention to this beautiful example of modern number theory.

42: [Democritus] Fragment 9, Sextus, *Adv. Math*. VII, 135 (Kirk and Raven, p. 410).

42: [Ampliatio] See the 1856 edition of Bouvier's *Law Dictionary,* ¶ 8 in the definition of "Postulatio".

44: [Magical but watertight] Working out the inductive step isn't always easy; it can at times be very hard, as in the first of Gauss's six proofs of the Law of Quadratic Reciprocity (see the Annex to Chapter Six).

44: [The empty form] Van Stigt, pp. 153, 306.

44: [Two-ity] Van Stigt, pp. 302–3.

44: [Maurolico] Details of Maurolico's life are from *The Dictionary of Scientific Biography,* ix, pp. 190–94; Kline, p. 223; St. Andrews website. See too G. Vacca, "Maurolico, the First Discoverer of the Principle of Mathematical Induction", in the *American Mathematical Society Bulletin* 16 (1909–10), pp. 70–73.

45: [Maurolico's proof for the sum of the first n odds] Kline, p. 272.

45: [Levi ben Gerson] Katz, p. 279; Stillwell, p. 136.

45: [Abu Bakr and others on early notions of induction] Katz, pp. 235, 238–41.

46: [Ibn al-Haytham and al-Samaw'al] Katz, pp. 238–39.

46: [The pearl of price] To us it seems odd that they prove a proposition for the values 1, 2, 3, 4, 5 and then say that you could go on in this way. Was the distinction between the potential and actual infinite, which will become so important in Chapter Nine, even more vivid to them than it is to us?

46: [Through Euclid] See Euclid VII,13.

46: [Peano's symbols and his students] Kline, p. 988.

46: [Peano's Axioms] Peano acknowledged in 1891 ("*Sul concetto de numero*", in *Rivista di Matematica*) that he had gotten his axioms from Dedekind's *Was sind und was sollen die Zahlen?* of 1885.

47: [Riemann] Imre Lakatos, in his *Proofs and Refutations* (Cambridge University Press, 1976, p. 9n) gives this quotation as: "If only I had the theorems! Then I should find the proofs easily enough." There is a sharp irony in this, since Riemann's great insight of 1859, known as the Riemann Hypothesis, is still unproven.

47: [*Psychology of Military Incompetence*] This delightful book is by Norman Dixon and was published by Jonathan Cape in 1976. The characteristics listed here are on p. 347.

47: [Language only touches the outside] Van Stigt, pp. 135–45, especially p. 144.

47: [Soul taken from its deepest home] Van Stigt, p. 137.

47: [Enforcing our wills via language] Van Stigt, p. 144.

47: [Truths fascinating by their immobility] Van Stigt, p. 25.

48: [Mathematics rooted in life] Van Stigt, pp. x and vii.

48: [Intuition from primordial elements] Van Stigt, p. xi.

48: [Limitless unfolding] Van Stigt, p. 157.

48: [Individual mind mattered] Van Stigt, p. x.

48: [Enrolled in 1897] Van Stigt, p. 23.

48: [Couldn’t stand others] Van Stigt, pp. 23, 32.

48: [Pilgrimages to Italy] Van Stigt, p. 26.

48: [Hut] Van Stigt, p. 35.

48: [Motley plurality] Van Stigt, p. 138.

48: [Eyes closed] Van Stigt, p. 45.

48: [As Weyl put it] Reid, p. 270.

48: [Xenophanes] From fragments 26 and 25, Simplicius *in Phys* 23: οὐδὲ μετέρχεσθαί μιν ἐπιπρέπει ἄλλοτε ἄλλῃ, ἀλλ᾽ ἀπάνευθε πόνοιο νόου φρενὶ πάντα κραδαίνει. “Fitting in” for Brouwer (Du. *inpassen)* is different in etymology and flavor from Xenophanes’s “fitting” = ἐπιπρέπει, but their images of the unmoved mover are remarkably similar. On Brouwer’s view of constructive fitting together via 1–1 correspondences, see van Stigt, pp. 240–41.

48: [Portrait of Brouwer] From a photograph in van Stigt, p. 51.

48: [Causal thinking is low cunning] Van Stigt, p. 31.

48: [“Go completely crazy”] Van Stigt, pp. 31, 32.

48: [Passage on air and mud-bathing] Van Stigt, p. 34.

49: [Lost friends] Van Stigt, pp. 107–9.

49: [Secretary] Van Stigt, p. 110.

49: [Long haired, lean and fit] See photograph in van Stigt, p. 108.

49: [Wordsworth on Newton] From *The Prelude,* Book III.

49: [Self-evident] It is worth noting that for Brouwer, ‘self-evident’ meant ‘meaningless’ (cf. Wittgenstein and ‘tautology’). See van Stigt, p. 157.

49: [Regress of justifications] The same dilemma confronted the Stoics in trying to explain what made their φαντασία καταλεπτική "evident". See Long (op. cit.), p.19.

49: [Tower of turtles] Turtles, tortoises, elephants, or some interleaving of them? See D. Panda, *The Rationale for Astrology* (Bhubaneswar).

49: [Portrait of Hilbert] From a photograph in Reid, p. 71.

50: [Hilbert and consistency of geometry] Reid, p. 64.

50: [Hilbert's plan for the new century] Reid, p. 71.

50: [Postulates and petitioning] Barnhardt's *Dictionary of Etymology* suggests that postulate "...probably borrowed from meaning in mediaeval Latin *postulatus*, ppp of *postulare* = to nominate to a bishopric. Also petition or request, from Latin *postulata* = things requested, from *poscere* = woo. I.E. *p*^{r}*k-ske* (therefore related to pray)."

50: [Free play] Reid, p. 63 quoting Max Dehn on Hilbert's *Foundations of Geometry*: "...the characteristic Hilbertean spirit... taking advantage to the fullest of the freedom of mathematical thought!"

50: [Plato on noblest games] *Laws,* vii.

51: [Poincaré on formalism] Reid, p. 63.

51: [Poincaré's own suggestion] Reid, pp. 99, 186.

51: [Brouwer on existence] Van Stigt, p. 133.

51: [Brouwer: a false theory is false] Kline, p. 1208.

51: [Brouwer on constructed truths alone existing] Van Stigt, pp. 266–67.

51: [Working the miracles of mathematics] The issue that no foray into or with language ever seems to resolve is how to explain existence. Aristotle had long since pointed out that defining something doesn't make it exist. This shows the fatal split of language away from ontology. Neither Formalism nor Intuitionism answers Leibniz's hollowly-resounding question: "Why is there something rather than nothing?"

51: [The Great Converse] This is an argument that has itself taken on a whole gamut of forms. What was "compossibility" in Leibniz is the "possible worlds" argument now. It is an argument you would think Cantor would have found congenial. Why, then, does he so rail against the suggestions made by Du Bois-Reymond, Stolz, and Vivanti that infinitesimals could be made via his cardinals? Because, it seems (see Dauben pp. 128–32), such would have threatened the simplicity of the real line, and his Continuum Hypothesis.

51: [Hilbert to Frege] This letter of Dec. 29, 1899, is quoted from Gregory H. Moore's "First-Order Logic as the Basis for Mathematics", in Phillips, pp. 109–10. Moore quotes from G. Frege, *Philosophical and Mathematical Correspondence* (ed. Gabriel et al, trans. H. Kraal, University of Chicago Press).

52: [Axioms begetting no contradictions] In 1840 Duncan Gregory wrote: "...the step which is taken from arithmetical to symbolic algebra is that... we suppose the existence of classes of unknown operations subject to the same laws." (Kline, p. 774). Hilbert's advice is to leave out of question what these operations—or the objects they are performed on—*are,* and try to guarantee by their mere form the consistency of the results of operations on symbols emptied of meaning.

52: [Brouwer walked out of a dinner] Reid, p. 187.

52: [Hilbert threw Brouwer off board] Reid, p. 187; van Stigt, p. 101.

52: [Conference in Bologna] Reid, p. 188; van Stigt, p. 101.

52: [Kant on mathematics from 'intuition'] *Prolegomena,* Part I, especially §10.

52: [Gauss spotting that Kant was wrong on details] Kline, p. 872.

52: [Hilbert's doctoral exam] Reid, p. 17.

52: [Hilbert's farewell address] Reid, p.194–95.

53: [Hilbert on intuitive insight] On the importance of intuition to Hilbert, see Reid, pp. 60, 62, 64, 184, 186, 194–96. In his great paper, "David Hilbert and his Mathematical Work" (*Bulletin of the American Mathematical Society* 50 (1944), pp. 612–54—reprinted in a shortened version in Reid), Weyl says tellingly: “[Hilbert] becomes strict formalist in mathematics, strict intuitionist in metamathematics.” Reid, p. 270.

53: [Hamilton and the intuition of time] Sir William Rowan Hamilton, “Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary Essay on Algebra as the Science of Pure Time”(1833,1835), in *The Mathematical Papers of Sir William Rowan Hamilton* (ed. H. Halberstam, and R. E. Ingram), (Cambridge University Press, 1967, vol. III, 3–99): pp. 5–6.

53: [Asymmetry took hold of Brouwer] Van Stigt, p. 159.

53: [Brouwer defended the remaining bastion] Van Stigt, p. 151.

53: [Intuition from passive stamp to active agent] Van Stigt, pp. 150–51.

53: [Primordial happening] Van Stigt, xi, pp. 147–53.

53: [Aware of existing in time] Mathematics ever anticipates philosophy. Brouwer’s thoughts about the Primal Happening long predate Heidegger and Sartre.

53: [Silent reflection] Van Stigt, p. 158.

53: [Two species of time] Van Stigt, pp. xi, 153–56.

53: [Germs caught from others] Van Stigt, p. 159.

53: [Constructional beauty] Van Stigt, p. 139, and see pp. 137–38, 143 and 124–25.

53: [Mathematics sinful] Van Stigt, p. 400.

53: [Functions worked accurately on numbers] Van Stigt, pp. 91–93, 379–85.

53: [Brouwer's fundamental theorem] The following passages, paraphrased from van Stigt (with page-references to him), spell out this theorem and its background more fully. (pp. 298, 381–84): The Fundamental Theorem of Finite Spreads (or Fan Theorem), along with the Bar Theorem, with which its proof begins, were conceived by Brouwer in 1924 as lemmas of the Uniform Continuity Theorem. The Fan Theorem states that if to every element e of a finite set M a natural number βe can be assigned, then a natural number z can be determined such that for every e, βe is completely determined by the first z choices generating e. Brouwer said that this is "... a wonderful theorem whose importance would justify to call it the fundamental theorem of intuitionism."

The Bar Theorem states: If to each element of a set M a natural number β is assigned, then M is split by this assignment into a well-ordered species S of point-sets, Mα, each of which is determined by a finite number of choices, and to each element of one Mα the same natural number βα is assigned. Its proof, says van Stigt, was metamathematical even in the Brouwer sense. (p. 298): The Uniform Continuity Theorem states that every function defined on the closed unit interval is uniformly continuous. It follows directly from the definition of "full function"and the Fan Theorem.

It may help to read what Weyl said in 1920 in defense of Brouwer's continuum, along with Brouwer's comments on his lecture (p. 379): "It is clear that one cannot explain the concept ‘continuous function in a bounded interval’ without including ‘uniform continuity’ and ‘boundedness’ in the definition. Above all, there cannot be any function in a continuum other than continuous functions. When the Old Analysis introduced ‘discontinuous functions’ it showed most clearly how far it had departed from a clear understanding of the essence of the continuum. What is nowadays called a discontinuous function is in reality no more than a number of functions in separate continua [Brouwer: “Better to say ‘the function is not everywhere defined’”]. Take for example the continua C, C^+ ($x > 0$) and C^- ($x < 0$)... If we consider the two functions +1 in C^+ and –1 in C^- then *there does not exist* a function defined for the whole C equivalent with the one value for C^+ and the other value for C^-.” [Brouwer: “Very fine! Underline because this is the main and most important point.”]

54: [Hopes lie scattered] Van Stigt, p. 298 on loss of confidence; p. 379 on proofs eluding him; pp. 93, 98 on unfinished work.

54: [Melancholia’s tools] The reference is to Dürer’s great engraving of 1514, “Melencholia I”.

54: [Silence] Van Stigt, pp. ix, 103–10.

54: [And the great fleas...] These lines of De Morgan’s follow those quoted on p. 16.

54: [Out-flanking maneuver] Gödel’s Theorems came as a shock—not only on general grounds, but because Hilbert had so triumphantly succeeded in proving the consistency and completeness of Propositional Calculus, a tidy domain of logic. While Gödel turned mathematics away from consistency proofs, these seem to linger on in physics, interestingly enough.

54: [Neith] In Proclus, *In Timaeum* I. 30. His description agrees with Plutarch's account (in his essay on Isis and Osiris) of the attributes of this very ancient Egyptian goddess of (it may be) hunting.

54: [Hilbert's last public words] Reid, p. 196.

THREE: DESIGNS ON A LOCKED CHEST

56: [Row] This may not have been exactly a row, and considerable opportunity beckons to find just the right word for it in the vocabularies of the people attuned to subtle distinctions in this area of human endeavor. Scots offers possibilities in 'ramie' and 'stuchie' (the latter, our informants tell us, is often heard but never written; the former—also 'rammy'—is in other circumstances 'linen and cotton'). We shouldn't forget 'shindy'. What is needed is not only a connoisseur's ear for shades of distinction but a connoisseur's eye for how much intellectual posturing here elbows out actual intellectual violence.

58: [J. B. Brown's poem] We give the first and last verses in the text. Here are the middle four verses:

> I have broadened and strengthened a mind
> With zeal ever burning more fiercely for learning

Of every conceivable kind.
In divers directions my knowledge—since college—
Has grown to be quite “omnibus”,
But I’m still more or less in the dark, I confess,
Why – *plus* – makes –
But – *times* – makes +

I have lapped up the learning of Livy,
Have battled with *Caesar* in *Gaul*,
To practice—in Attic—the Method Socratic
Does not incommode me at all;
I have plumbed all the pages of Plato and Cato
And Varro and Vergilius,
But I’m forced to admit that I can’t see a bit
Why – *plus* – is –
But – *times* – is +

The Aristotelian viewpoint
The critical croakings of Kant,
The racy remarks of both Engels and Marx,
And Lubbock, and Fabre on The Ant,
Pirandello and Browning’s odd fellow “Sordello”,
I’m fully equipped to discuss,
But I’m sorry to say that I’m still not *au fait*
Why – *plus* – is –
But – *times* – is +

I have read all the writings of Rousseau,
I have mastered the musing of Mill,
I find joy unalloyed in the fancies of Freud
And the carvings of Epstein and Gill;
I am strong on the coaching of Cotton and hot on
The heresies horrid of Huss,
But although I still try, I can't understand why
A – *plus* – is –
But – *times* – is +

"Limitation" is in *Punch* vol. CXC, March 25, 1936, p. 350. Would Brown have been helped—would you?—by this sonnet of Thomas Harriot's, written three centuries before him:

If more by more must needs make more
Then lesse by more makes lesse of more
And lesse by lesse makes lesse of lesse
If more be more and lesse be lesse

Yet lesse of lesse makes lesse or more
Use which is best keep both in store
If lesse of lesse you will make lesse
Then bate the same from that is lesse.

But if the same you will make more
Then add to it the signe of more.
The rule of more is best to use
Yet for some cause the other choose.

So both are one, for both are true
Of this inough and so adieu.

Fauvel and Gray (from p. 292 of whose anthology this comes) remark that Harriot took great pains over this sonnet. Certainly its readers do.

58: [Proof that (–a)(–b) = ab] William Rowan Hamilton tried to embody this argument when he made his plea for taking the intuition of time as the source of mathematics. He had already freed himself from our accustomed "number *line"* during his traffic with complex numbers, which broadened out to a plane (as you will see in Chapter Seven). Here that freedom took a different form (the following paraphrases Hamilton, op. cit., pp. 27–28):

Time flows inevitably from past to future, through the present. Take "–" to mean reversing your direction in time (once again, mathematics is freedom—even from such an iron necessity as this). So "5" means moving forward five units from the present, 0, to the future. "–5" means moving backward from 0 to the past (you might say "5 ago"). If you have this 'reversing of direction of time' well in mind as the meaning of "–" (call time machines to your aid if you wish), what comes next will be clear.

(+3)•(+4) means taking a length 4 units of future time long and tripling it forward in time: so to +12.

(+3)•(–4) means taking the –4 length—i.e., 4 units in the past ("4 ago") and tripling it; so back to –12.

(–3)•(–4) means taking that –4 length, then *reversing your direction in time* (this is –3, not +3), hence toward the future. Tripling brings you to +12.

Hamilton was a classicist as well as a mathematician. He would have been thoroughly familiar, from Greek and Latin as well as English, with the tense called “future perfect”: putting yourself in the position of future time and looking back at an event which would *then* be past, although from the point of view of the present, might still be future (“You will have understood this explanation by the time it is over”). Perhaps this gave him the background for his image.

59: [Undeserving A and meritorious B] They appear as such in Gilbert and Sullivan’s *Mikado,* in the song “See How the Fates Their Gifts Allot”, sung by Pitti-Sing, Pooh-Bah, and the Mikado himself.

60: [Euclid’s floruit] Heath, i, p. 354.

61: [Euclid’s proof] Euclid, IX, 20.

61: [Erdos on believing in The Book] Quoted in the preface to Martin Aigner, and Günter M. Ziegler *Proofs from the Book* (Springer, 1999).

61: [Portrait of Erdos] From a photograph (one of his mother’s favorites) in Paul Hoffman’s *The Man Who Loved Only Numbers* (Fourth Estate, 1999).

61: [Date of Eratosthenes] Heath, ii, p. 104.

62: [A beta mind] Heath, ii, p. 104.

65: [Octillion] The truth of these statements holds whether you reckon your octillions American style (10^{27}) or by the more generous British measure (10^{48}).

66: [Leopardi] The translation by John Heath-Stubbs is from his *Giacomo Leopardi: Selected Prose and Poetry* (Oxford University Press, 1966). The original is:

L'Infinito

Sempre caro mi fu quest'ermo colle,
E questa siepe, che da tanta parte
Dell'ultimo orizzonte il guardo eclude.
Ma sedendo e mirando, interminati
Spazi di là da quella, e sovrumani
Silenzi, e profondissima quiete.
Io nel pensier mi fingo; ove per poco
Il cor non si spaura. E come il vento
Odo stormir tra queste piante, io quello
Infinito silenzio a questa voce,
Vo comparando: e mi souvien l'eterno,
E le morte stagioni, e la presente
E viva, e il suon di lei. Così tra questa
Immensità s'annega il pensier mio:
E il naufragar m'è dolce in questo mare.

67: [Biographical details of Dirichlet] From the St. Andrews website.

67: [Russians and time-tables] In Yevgeny Zamyatin's *We*, the *Time-Tables of All the Railroads* was "the greatest of all the monuments of ancient literature." Certainly Nabokov testifies to this obsession (as in the first pages of his

Pnin), and any reader of English country-house mysteries knows the importance of the shadowy Bradshaw.

68: [Portrait of Gauss] Based on J. B. Listing's sketch and Bendixen's 1828 portrait.

69–73: [Graphs] Zagier, pp. 9, 10.

72: [Where Li(x) and π(x) cross] Zagier, p. 18.

72: [Prime-free gaps] Zagier, pp. 11–12.

73: [Quotation from Zagier] Zagier, p. 8.

73: [Palindromic, counting, and topping and tailing primes] For these and other such, browse through David Wells's *The Penguin Dictionary of Curious and Interesting Numbers* (Penguin Books, 1987).

73: [Modern work on twin primes, and 1.3a] Zagier, p. 17, note 9.

74: [Autists] Consider this passage from Oliver Sacks's *An Anthropologist on Mars* (Vintage Books, 1995, pp. 269–70): "I was struck by the enormous difference, the gulf between Temple's immediate, intuitive recognition of animal moods and signs and her extraordinary difficulties understanding human beings, their codes and signals, the way they conduct themselves... When she was younger, she was hardly able to interpret even the simplest expression of emotion; she learned to 'decode' them later, without necessarily feeling them... Lacking [implicit knowledge of cultural presuppositions], she has instead to 'compute' others' intentions and states of mind, to try to make algorithmic, explicit, what for the rest of us is second nature."

INTERLUDE: THE INFINITE AND THE INDEFINITE

75: [Solon] Fr. 24, Diehls, lines 5–7.

75: [Anaximander and the apeiron] Kirk and Raven, pp. 105–17.

75: [Onlie begetter] The original onlie begetter was, of course, Shakespeare's Mr. W. H.

76: [da Vinci] "*Di mi se mai fu fatto alcuna cosa.*" Quoted in Kenneth Clark's wonderful essay "The Concept of Universal Man", in his *Moments of Vision* (John Murray, 1981), p. 98.

FOUR: SKIPPING STONES

77: [Passage from Newton] Bell, p. 90.

79: [Weil and proven conjectures] Weil was, of course, as alive as anyone to a conjecture's implications—all the more so, once it was proven. The theorem itself and its proof, however, no longer interested him. "One achieves knowledge and indifference at the same time," he wrote to his sister Simone (*André Weil Oeuvres scientifiques* [Springer, 1979] i, pp. 244–55, *"Une lettre et un extrait de lettre Simone Weil"*), in what seems a parody of Buddhist doctrine. See his *“De la metaphysique en mathematique”* (ibid., ii, pp. 408–12).

82: [Eureka! num = Δ + Δ + Δ] Gauss’s diary for July 10, 1796. From Bell, p. 228.

85: [Hobbes and Wallis] Quoted in Fauvel and Gray, p. 316, from Hobbes, “Six Lessons on the Professors of Mathematics” (*Collected Works,* VII, 1839–45, pp. 315–16), and J. Wallis, “Due Correction for Mr. Hobbes, or Schoole Discipline, for not Saying His Lessons Right” (1656), p. 50.

88: [The elusive she] She was once spotted by Swann, disappearing in a crowd; prior to that, by Goethe, in the form of the *ewig Weibliche*.

88: [On the way to Moscow] This mood of always being on the way but perhaps never getting there seems built into Russian verbs, in their imperfective aspect—as so charmingly explained in Alexander Lipson’s *A Russian Course,* which metonymously never quite made it to publication (the preliminary edition appeared in 1968 under the imprint of Slavica Publishers Inc., Cambridge, Mass.).

88: [Brouwer’s fundamental sequence] Van Stigt, pp. 305, 312.

89: [For Brouwer, objects are sequences of sequences] Van Stigt, p. 189.

90: [Euclid’s differently elegant proof] Euclid, IX, 35.

91: [Visual proof] This wonderful proof is due to J. H. Webb, in Roger B. Nelsen's *Proofs Without Words* (Mathematical Association of America, 1993), p. 119.

91: [Critics of proof by picture] James Robert Brown, in the last chapter of his *Philosophy of Mathematics* (Routledge, 1999), collects and criticizes positions taken against visual proofs, which include: an apparent lack of rigor; mathematics seen as arising essentially from the manipulation of verbal or logical symbols; and pictures as a source of error (although he doesn''t mention that algebraic manipulations can make transparent what pictures—as in knot theory—may obscure). He tellingly quotes (p. 173) Pierre Cartier, a member of Bourbaki (the group of primarily French mathematicians who vigorously promulgated rigor in the mid-twentieth century), who, when asked why there were no diagrams in Bourbaki, answered: "The Bourbaki were Puritans, and Puritans are strongly opposed to pictorial representations of truths of their faith."

91: [Littlewood] *A Mathematician's Miscellany* (Methuen, 1953), p. 35, on the diagram illustrating that if f(x) is an increasing function, then for any x in [0,1] the sequence x, f(x), f(f(x)), ... has a fixed point.

92: [Powers read as greater because they work with the unembodied] You could read the ancient edict against graven images as Freud did, in his otherwise discredited *Moses and Monotheism*, a call to thought which is deeper because more abstract. Even in cultures as tied to the visual as the Greek, you find Aristotle in effect warning that the diagrams accompanying proofs are only aids to memory (*De Memoria,* 450a). Certainly memory plays an interestingly intermediate role in recursive abstraction (as memory palaces testify): somewhere between the visual and the structural, and curiously out of time.

92: [On Nicole d'Oresme] Kline, p. 437; Stillwell, p. 119.

93: [Toricelli's trumpet] The volume of this infinite trumpet is, astonishingly, π.

93: [Toricelli and Oresme] The connection between Oresme's tower and Torricelli's trumpet was pointed out by John Stillwell, p. 119. There too (p. 103) is the Hobbes quotation from his "Considerations upon the Answer of Dr. Wallis".

93: [Peacock's third impossible thing] Kline, p. 974.

93: [Peacock's fourth impossible thing] Kline, pp. 974–75.

94: [Laws in the wilderness] The comparison to statute and common law might be worth pursuing. We tend to invent our axioms only when need requires, as in common law—and then come to see them as having an aura even more golden than do statute laws—that is, as pro- rather than re-active.

94: [Nicole d'Oresme and the harmonic series] Kline, p. 437.

95: [Minkowski anecdote] Reid, p. 102.

95: [Portrait of Minkowski] From an unattributed photograph on the St. Andrews website.

96: [Poincaré and logical monsters] Kline, p. 973.

96: [Abel on divergent series] Kline, p. 973.

96: [1 – 1/2 + 1/3 –.. converges] Entertainingly enough, it converges to ln 2, an irrational that begins .693...

98: [The tower of xs] Be careful when trying this at home with numbers other than 2 to the right of the equals sign. The expression

$$x^{x^{x^{x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}}}}} = y$$

turns out to be meaningful only when y lies between 1/e and e:

$$1/e \leq y \leq e$$

that omnipresent e, the base of the natural logarithms, which we have now met for the third time.

FIVE: EUCLID ALONE

100: [Graves] *Goodby to All That* (London: Cassell and Co., 1958), p. 260.

100: [Euclid’s Fifth Postulate] What Euclid has is: “If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if indefinitely produced, will meet on that side on which are the angles less than the two right angles." The form given in the text, known as Playfair's Axiom, is equivalent to Euclid's. By putting the matter negatively Euclid may have been trying to do an end-run around the infinite.

101: [Greek uneasiness with fifth postulate] Euclid invoked the infinite (τὸ ἄπειρον) not only in the fifth but in earlier postulates, as in granting that we may construct a line through *any* two points and a circle with *any* radius. As Heath observes (Euclid, i, p. 200), "The circle may be indefinitely large, which implies the fundamental hypothesis of *infinitude* in space. This... is essential to the proof of I.16 [the exterior angle of a triangle is greater than either opposite interior angle], a theorem not universally valid in a space unbounded in extent but finite in size." It wasn't just invoking the infinite in the fifth postulate that disturbed his contemporaries and successors, but Euclid's insistence on assuming what they thought should and might yet be proved (his genius, says Heath, is shown by his choice). For as Proclus remarked (Euclid, i, p. 203), asymptotes showed that certain converging lines won't meet, so we really need to prove that converging straight lines must.

103: [Pythagoreans and a triangle's angle-sum] Heath, i, p. 135.

104: [Hobbes] Aubrey's *Brief Lives*, ii, pp. 220, 221.

114: [Archimedes looked to physics for his insights] Heath, in his excellent edition of Archimedes' *Method* (*The Works of Archimedes* [Dover reprint of the 1897 Heath edition, with 1912 Supplement], says that the mechanical method used by Archimedes "...and shown to be so useful for the discovery of theorems is distinctly said to be incapable of furnishing proofs for them; and Archimedes promises to add... the necessary supplement in the shape of the formal geometrical proof.” (p. 7).

114: [Two early results from Euclid] That the line joining the midpoints of a triangle’s sides is parallel to its base follows from VI, 6 and I, 28; that this line is half the length of the base, from VI, 6 and VI, 4. The second result—that the diagonals of a parallelogram bisect each other, follows from I, 4, 8, 26, 29, and 34.

119: [Portrait of Euler] From an unattributed portrait on the St. Andrews website. For details of his life, see Stillwell, pp. 132–34 and Kline, pp. 401–3.

120: [Thales’s theorem on an inscribed triangle] Heath, i, p. 131; on Pamphile, i, p. 133 (her date under Nero, 54–68 A.D.).

121: [Feuerbach] A recluse: St. Andrews website. See too Kline, p. 837. This proof was first published in 1821 by Gergonne and Poncelet.

124: [Euclid alone] Millay‘s sonnet (from *The Harp-Weaver*) reads:

> Euclid alone has looked on Beauty bare.
> Let all who prate of beauty hold their peace,
> And lay them prone upon the earth and cease
> To ponder on themselves, the while they stare
> At nothing, intricately drawn nowhere
> In shapes of shifting lineage; let geese

Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

124: [*L'art de bien raisonner…*] The passage occurs in Poincaré's article "Analysis Situs," *Journal de l'école polytechnique* (1895) t.1, pp. 1–121. It can also be found in volume vi of his *Oeuvres* (Paris, 1953), p. 194.

124: [Portrait of Poincaré] From an unattributed photograph in Stillwell, p. 310.

124: [Fagnano] Kline, p. 413.

126: [Polonius's advice to Laertes] *Hamlet,* ii.1. 66.

126: [Rhind papyrus] Katz, pp. 3, 13–14.

127: [Fejér] St. Andrews website.

INTERLUDE: LONGING AND THE INFINITE

131: [Saladin's army] Cited in Erik Durschmied's *The Hinge Factor* (Arcade, 2001) p. 13. In modern set theory 'uncountable' can mean simply that the set cannot *within the model* be put into a 1–1 correspondence with the counting numbers.

131: [Crowds at football stadia] Michigan, Tennessee, and Penn State can accommodate over 100,000 fans; 98,000 can watch rivetting cricket in Melbourne.

131: [All the leaves on all the trees] This echoes two lines of Charles Elton's "Luriana Lurilee," quoted to such effect in Virginia Woolf's *To the Lighthouse.*

131: [Kenneth Clark] "Iconophobia", in his *Moments of Vision* (op. cit.), p. 33.

131: [The expansion of infinite things] *"Ayant l'expansion des choses infinies"*: from Baudelaire's *"Correspondances".*

132: [Michael Atiyah] "Mathematics in the Twentieth Century", *MAA Monthly* (August–September 2001, vol. 108, no. 7) p. 659.

132: [Aspiring to the condition of mathematics] This is meant to echo Walter Pater's "All art constantly aspires towards the condition of music." (in his essay "The School of Giorgione", in *The Renaissance*).

SIX: THE EAGLE OF ALGEBRA

133: [Title] In the background of this title lies a passage from J. J. Sylvester's "On a new Class of Theorems in Elimination Between Quadratic Functions", *Philosophical Magazine* XXXVII (1850) pp. 363–70 (also in Sylvester's *Collected Works,* i, pp. 145–51): "Aspiring to these wide generalizations, the analysis of quadratic functions soars to a pitch from whence it may look proudly down on the feeble and vain attempts of geometry proper to rise to its level or to emulate it in its flights."

133: [God and compasses] For a study of the theme of God holding a compass, See J. B. Friedman's "The Architect's Compass in Creation Miniatures of the Later Middle Ages", *Traditio, Studies in Ancient and Medieval History, Thought and Religion* (New York, 1974), pp. 419–29.

133: [Euclid and subtle devices] Katz, pp. 61–62.

134: [Lobkowitz] This enticing information comes from James Franklin's *The Science of Conjecture: Evidence and Probabiliity Before Pascal* (Johns Hopkins, 2001), p. 89.

137: [Childe Roland] He, as Child Rowland, and his tower appear in *Lear,* iii. 4. 187, then in Robert Browning's "Childe Roland to the Dark Tower Came".

137: [Golden ratio] The term "golden ratio" or "golden mean" may first appear in Kepler (passage quoted on p. 138). It is the "divine proportion" in Luca Pacioli's *De divina proportione*, published in 1509.

137: [Euclid's definition] This appears as Definition 3 in Book VI of the *Elements*.

138: [The beauty of the major sixth] See H. E. Huntley, *The Divine Proportion* (Dover, 1970), pp. 51–53.

138: [Kepler on extreme and mean ratio] Kepler, *Harmonices Mundi.*

138: [Pentagram called “Health”] Heath, i, 161; Euclid, ii, 99.

141: [Hippasus and the incommensurable] Was it the irrationality of $\sqrt{2}$ or of ϕ, the golden mean, that Hippasus is supposed to have discovered? This is the kind of grist that keeps the mills of scholarship grinding slow but exceeding fine. Such scanty evidence as we have is indirect or from untrustworthy sources.

Kurt von Fritz, in his striking essay “The Discovery of Incommensurability by Hippasus of Metapontum” (*Annals of Mathematics,* no. 46 (1954) pp. 249–64, reprinted in David J. Furley and R. E. Allen (eds.), *Studies in Presocratic Philosophy* (Humanities Press, 1970–1975, I, pp. 382–412), says (p. 386) that tradition is unanimous in attributing the discovery of the former to Hippasus, and brings indirect evidence to bear from Plato’s *Theaetetus* and the unreliable Iamblichus. Heath (i, p. 27) asserts that the proof was “doubtless” the one we gave in our first chapter.

Von Fritz declares that Hippasus devised a proof that the side and diagonal of a pentagram were incommensurable, tracing his approach back perhaps centuries before the mid-fifth century B.C. to a rule of thumb well known to craftsmen. For if you wish to find the greatest common measure of two lengths s and l, simply lay the shorter one s off on l until no or some remainder $0 < r_1 < s$ is left. If none remains, s is their greatest common measure. Given r_1, however, lay *it* off in turn on s until some r_2 with

$0 \leq r_2 < r_1$ remains. If $r_2 = 0$, r_1 is the greatest common measure; if not, continue in this way, obtaining successively smaller remainders r_3, r_4, ... until the process terminates—and when it does, that last remainder is the greatest common measure of s and l.

Yet what if the process *never* ends? Then indeed s and l are incommensurable, and if s, say, is an integer or rational, l must be irrational. How could you prove, however, that the process would never end? Here is where the brilliant insight appears. If each "laying off" operation is similar to the previous one, then the process must continue forever. Now when you draw in all the diagonals of a pentagon, another, similar and smaller pentagon appears upside-down within it. By isosceles triangles, the diagonal of this inner pentagon will be congruent to the remainder $r_1 = l - s$ of the original pentagon; hence the same process will yield an r_2, and so on.

The pentagons may shrink in size but by similarity can never disappear—hence the sequence of remainders continues forever and s and l are incommensurable. Their ratio ϕ is irrational.

Scholarly grinding pulped von Fritz's paper as soon as its ink was dry. So J. A. Philip, in *Pythagoras and Pythagoreanism* (University of Toronto Press, 1966), finds his thesis "untenable" (p. 30) because his sources are notoriously corrupt. For a subsequent defense of von Fritz, see D. H. Fowler, *The Mathematics of Plato's Academy* (Oxford: Clarendon Press, 1999).

And Hippasus? He may have been drowned by the Pythagoreans or by the gods themselves for his impiety in discovering—or perhaps making public the discovery—of the irrational; or he may have been the figment of an ancient imagination; or a peg (as Philip says) on which historians of mathematics have hung their hypotheses. He is certainly farther out than we thought in the turbulent waters of academic debate.

149: [*Piers Plowman's Vision]* These lines are from the Induction, 17–18, as translated by Henry W. Wells.

149: ["Put them to the plough..."] *Piers Plowman*, lines 19–21.

150: [Making a quotient tell us its name] This technique—called multiplying by the conjugate a – b√c of a number a + b√c—is in Euclid X, 112, although done geometrically, in terms of the areas of rectangles. It may, says Heath, stem from the later work of Apollonius. Heath comments on this proposition: "The proof is a remarkable instance of the dexterity of the Greeks in using geometry as the equivalent of our algebra. Like so many proofs in Archimedes and Apollonius, it leaves us completely in the dark as to how it was evolved. That the Greeks must have had some analytical method which suggested the steps of such proofs seems certain; but *what* it was must remain apparently an insoluble mystery." (Euclid, ii, p. 246).

153: [Glimmerings of the coordinate plane] Smith (ii, p. 316) sees the Egyptian glimmerings in their surveying practices, pointing out that their hieroglyph for a district (hesp) was a grid. Among the Greeks, he says, Hipparchus (c. 150 B. C.) used longitude (μήκος) and latitude (πλάτος) to locate earthly and celestial objects. Stillwell (p. 16) points out that Nicole d'Oresme in the fourteenth century took a step beyond the Greeks by setting up something of a coordinate system before determining a curve—but it took the algebraic skills and insights of Fermat and Descartes, working independently on a problem of Apollonius, fully to develop the coordinate plane.

157: [Passage from Wellington] quoted in Dixon, op. cit., p. 324.

161: [Princess Ida] These lines are from the song sung by Arac, Guron, and Scynthius, in Act III.

162: [“Here may we sit...”] From Samuel Daniel’s “Ulysses and the Siren”.

163: [Passage from Gauss] Quoted in Goldman, p. 203.

164: [Passage from Gauss] Fauvel and Gray, p. 492; translation by J. Gray.

165: [Irreducible cube or higher roots] If the number of sides of a polygon is a prime p, then, Gauss wrote, “as often as p – 1 contains other prime factors besides 2, we arrive at higher equations, namely to one or more cubic equations if 3 enters once or oftener as a factor of p – 1, to equations of the fifth degree if p – 1 is divisible by 5, etc.” (Kline, p. 753).

165: [Remark of curator of models] In a personal communication from S. J. Patterson, August 23, 2001. That a dissertation on a mathematical topic should be so little read is no surprise; that it should have been so much looked at, is. Alf van der Poorten (*Notes on Fermat’s Last Theorem* [Wiley-Interscience 1996], p. 42) says: “It appears that the average [mathematical] paper is read by some 0.76 mathematician, including author, referee, and reviewers,” and in a footnote adds a “...dictum that one’s Ph. D thesis should be readable by at least two people, one of them by preference being the author.”

165: [Ideal palaces] The *Palais Idéal*, built over many years by *le facteur* Cheval of Hauterives, south of Louhans—*vaut le voyage,* especially in the rain.

166: [Palmanova] This geometric abstraction materialized thirteen miles from Udine, in the Friulian plain, in 1593, to keep off the Turks and Austrians. The complicated fields of fire afforded by its nine bastions seem never to have sprouted a single battle.

166: [Puzzle—or problem?] The reduction of philosophy's problems to the puzzles of a game (or is it the elevation of puzzles to philosophical problems?) runs like a red thread through the later writings of Wittgenstein, unifying their apparent diversity. From *Philosophical Investigations*, p. 109 (trans. Anscombe): "... [philosophical] problems are solved, not by giving new information, but by arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of language."

SEVEN: INTO THE HIGHLANDS

167: [De Morgan] Kline, p. 975, quoting from his *Differential and Integral Calculus.*

167: [Calgacus] Tacitus, *Agricola* 30. As Simon Shama points out in his history of Britain: in keeping with the Roman tradition of historical writing, Tacitus undoubtedly put these words into Calgacus's mouth.

167: [Writers in exile] All writers, of course, are in exile, since that's what it means to write. It is this combination of the vivid and the abstract that makes autobiography so enthralling. The French understood this well, calling their historic past tense the *passé simple* because it enhances poignancy through distancing.

168: [Passage from Hamilton] Hamilton (op. cit.), p. 3.

169: [Wallis and passage paraphrased from him] David Eugene Smith, *A Source Book in Mathematics* (McGraw Hill, 1929), pp. 46–48.

170: [Bombelli] Fauvel and Gray, p. 264.

170: [Cardano] For his life, see p. 295 and the note to it.

170: [Bombelli's ideas] See Federica La Nave and Barry Mazur, "Reading Bombelli" in *The Mathematical Intelligencer* (January 2002), pp. 12–26.

173: [d'Alembert] Kline, p. 595; Bell, p. 156; Stillwell, p. 201–2.

173: [Portrait of d'Alembert] From an unattributed engraving in Stillwell, p. 201.

173: [Historian on Virginia] David Hackett Fischer and James C. Kelly, *Bound Away: Virginia and the Westward Movement* (University Press of Virginia, 2000), p. 14 (it is also here that the quotation from Sir Humphrey Gilbert appears).

174: [The ingenuity of Wallis] Wallis says: "...where √ implies a Mean Proportional between a Positive and a Negative quantity" (Smith, ii, pp. 263–64). He doesn't state explicitly that his axes represent that mean proportional geometrically, but the conclusion is unmistakable.

174: [Thurston] William P. Thurston, "Mathematical Education", *Notices of the American Mathematical Society* (37: 7, September 1990), pp. 55–60.

174: [Euler equating two expressions] Kline, p. 629, who points out that Cotes, De Moivre, and Vandermonde probably made the same identification.

175: [Wessel] St. Andrews website: Caspar Wessel.

175: [Wessel's paper] Kline, p. 629.

177: [Argand] St. Andrews website: Argand.

177: [Servois on algebra] St. Andrews website: Argand.

178: [Paul Halmos] "The Heart of Mathematics", *The American Mathematical Monthly* 87 (1980), pp. 519–24.

178: [Hilbert passage] Reid, p. 81.

180: [Wessel, Argand, Euler on length of product vector] Kline, p. 629.

181: [Hipparchus, Menelaus and Ptolemy] Kline, pp. 119 ff.

181: [Etymology of sine] Following Smith (ii, pp. 615–16), the contorted history of the word for this smooth curve begins in India, where around 510 Āryabhaṭa called it "chord-half" (*jyā-ardhā*), abbreviated *jyā*. This turned in Arabic hands into *jîba*, and since only the consonants "j b" were written, it also came to be read by later Arabic writers as "*jaib*", "bosom, breast or bay". Hence in 1150 Girardus of Cremona translated the Arabic into "*sinus*": "bosom, bay, curve, fold of toga about the breast, land about a gulf, fold in land."

184: [Goethe passage] Passage 813 in Robert Edouard Moritz, *On Mathematics and Mathematicians* (Dover, 1958), translated from Goethe, *Maximen und Reflexionen*, Sechste Abtheilung.

187: [Visual proof of the sine and cosine addition laws] This nifty proof is due to Roger B. Nelsen, in his *Proofs Without Words II: More Exercises in Visual Thinking* (Mathematical Association of America, 2000), p. 46.

188: [Wittgenstein on mathematics] See, for example, *Tractatus Logico-Philosophicus* 6.22.

188: [The concealments of Archimedes and Newton] See Kline, p. 595; Stillwell, pp. 188, 194. On Archimedes in particular, consider this, from Heath's introduction to his translation of Archimedes' *Method* (op. cit. p. 6): "Nothing is more characteristic of the classical works of the great geometers of Greece, or more tantalizing, than the absence of any indication of the steps by which they worked their way to the discovery of their great theorems." On Newton, see J. M. Keynes's fascinating "Newton, The Man", reprinted in *The World of Mathematics*, ed. J. Newman (Simon and Schuster, 1956), i, pp. 277–85, in which he says: "Certainly there can be no doubt that the peculiar geometrical form in which the exposition of the *Principia* is dressed up bears no resemblance at all to the mental processes by which Newton actually arrived at his conclusions." (p. 279).

188: [Gauss in 1825 and 1831] Kline, pp. 631–32.

189: ["No great thing comes without a curse"] Sophocles, *Antigone,* 613 (οὐδὲν ἕρπει θνατῶν βιότῳ πάμπολύ γ' ἐκτὸς ἄτας).

191: [Indian mathematicians] Stillwell, p. 120.

191: [Newton's discovery of trigonometric polynomials] Stillwell, p. 108.

191: [Exponential series discovered by Newton] Kline, p. 438.

192: [Euler's boldness] Stillwell, p. 221.

192: [Gauss and Cauchy] Gauss clarified the meaning of complex numbers in 1811 (Stillwell, p. 221 and see Kline, ch. 27 passim); on Cauchy's paper of 1851, see Kline, p. 642.

193: [Benjamin Peirce] Edward Kasner and James Newman, *Mathematics and the Imagination* (Simon and Schuster, 1940), p. 104.

194: [de Moivre] St. Andrews website: De Moivre.

194: [Cotes and De Moivre] Stillwell, pp.193–95.

199: [Passage from Blake] Quoted in Kenneth Clark (op. cit.), p. 9.

EIGHT: BACK OF BEYOND

202: [Alberti] Alberti's dates are 1404–1472. The passage is from his *De Pictura* (ed. Cecil Grayson), Book I ¶31: "*Id istiusmodi est: velum filo tenuissimo et rare textum quovis colore pertinctum filis grossioribus in parallelas portiones quadras quot libeat distinctum telarioque distentum. Quod quidem inter* *corpus repraesentandum atque oculum constituo, ut per veli raritas pyramis visiva penetret.*" Although perspective drawing was first described by Alberti and perfected by Piero della Francesca, it probably originated with Brunelleschi. See Panofsky, "Dürer as Mathematician", in Newman (op. cit.) i, p. 605.

204: [Poncelet] Details of his life from H. Tribout, *Un grand savant: Le général Jean-Victor Poncelet* (Paris, 1936); and J. Bertrand, "*Eloge historique de Jean Victor Poncelet*", in *Eloges académiques* (Paris, 1890), pp. 105–29. Stillwell (p. 83) points out that it was Poncelet who introduced the line at infinity, and Kline (p. 842) says that he was the first fully to appreciate that projective geometry was a new branch of mathematics.

204: [Portrait of Poncelet] From an unattributed photograph on the St. Andrews website.

205: [Quotation from *Hamlet*] Is it Hamlet whom Tribout echoes in his life of Poncelet (op. cit.) when he writes of his stay in prison: *"Car que faire en un gîte, à moins que l'on ne songe?"*

205: [Two millennia of attempts to prove the parallel postulate] See the excellent discussion in Euclid, i, pp. 202–20.

205: [Saccheri and Lambert] In addition to the discussion in Heath's *Euclid* (see previous note), on Saccheri, see Kline, p. 866 and Stillwell, p. 257. On Lambert, see Kline, p. 868 and Stillwell, p. 258.

205: [Gauss on the ‘shameful part’] On the ‘*partie honteuse’*, see Jeremy Gray, “The Discovery of Non-Euclidean Geometry”, in Phillips, p. 45.

206: [Pencil] For the origin of this term, see Desargues, *Oeuvres* (ed. Poudra), (Paris, 1864) i, pp. 135–36, §4. It is worth remarking that ideals as pencils of lines apparently gave Hilbert his idea for proving consistency and completeness by adjoining ideal numbers, each standing for infinitely many numbers—and so getting around Kronecker’s ban on the infinite by being utterly finite in method. See Reid, p. 269.

206: [Without having to peer through a veil] Cf. Corinthians 3: 13–18.

210: [5000 rubber threads in a Kooshball] This information appears on the web at www.drtoy.com.

211: [“Koosh” from the sound of ball landing] See entry “Kooshball”on www.yesterday-land.com. For more on its nature and history, see www.genuineideas.com, www.bodfods.com, www.cni.org (which explains that the copyright office refused registration of “Kooshball” because its tactility could not be considered in judging its creativity), and isaac.exploratorium.edu (where you will learn that, as Newton predicted, a spun Koosh has, like a planet, an equatorial bulge).

213: [Duality] Katz, p. 786 points out that Poncelet used the Principle of Duality as a tool but never established it as a theorem.

213: [Passage from Keyser] In Moritz (op. cit.), passage 1880.

215: [Desargues] (1591–1661). Son of a well off Lyonnais family, he wrote on stone-cutting and sundials, designed an elaborate spiral staircase, an ingenious pump and the beginnings of projective geometry. A crater on the moon and a street in Paris are named for him. See the St. Andrews website.

221: [Haldane] According to Kenneth Clark (*Civilization,* ch. 13), what Haldane did say was: “My own suspicion is that the universe is not only queerer than we suppose, but queerer than we can suppose.”

223: [Hilbert’s remark] Reid, p. 31.

226: [Hilbert on Pappus’s Theorem] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (Chelsea, 1952) p. 132. The implicational relations among the key theorems of the projective plane are these: The Fundamental Theorem (along with axioms P1 – P4, of course) implies Pappus’s Theorem, which implies Desargues’s Theorem; and conversely, all of these taken together imply the Fundamental Theorem. See Robin Hartshorne’s *Foundations of Projective Geometry* (Benjamin, 1967), ch. 5, for a lucid exposition.

227: [Principle of Continuity] Kline, p. 843.

NINE: THE ABYSS

228: [Nietzsche] *Beyond Good and Evil,* Aphorism 146: *“Und wenn du lange in einer Abgrund blickst, blickt der Abgrund auch in dich hinein.”*

228: [A little cloud] I Kings 18:44.

228: [Thabit ibn Qurra] See A. I. Sabra, “Thabit ibn Qurra on the Infinite and Other Puzzles”, in *Zeitchrift für Geschichte der Arabisch-Islamischen Wissenschaften*, 11 (1997), pp. 1–33.

228: [Galileo] *Two New Sciences:* Day One (pp. 31–33 in the Dover edition).

229: [Glib know-nothingness] This is the *ignorabimus* which Hilbert said doesn’t exist in mathematics.

229: [Letter from Cantor’s father] This is condensed from the letter Georg Waldemar Cantor wrote to his son in 1860 (Dauben, pp. 274–76).

229: [Secret voice] Dauben, pp. 277, 283, 289–91.

229: [Portrait of Cantor as a young man] From a photograph in Purkert, p. 30, titled *“Cantor am Beginn seiner Hallenser Zeit”.*

232: [Nomads in the Caucasus] For this passage see *The Epic Histories Attributed to P’awstos Buzand* trans. and commentary Nina Garsoian, (Harvard University Press, 1989), Book III, ch. 7, p. 73. “They themselves could not number their own army,” wrote Buzand. Counting was and remains the most formidable of human undertakings.

232: [Cairn of Remembrance] On the *Cairn na Cuimhne* see Charles Maclean, *Romantic Scotland* (Cassell, 2001), p. 65.

236: [Aristotle, Gauss and Kronecker on completed infinities] Aristotle: Dauben, pp. 122–25. See, for example, *Meta.* IX. vi. 5. Gauss, writing to Schumacher on July 12, 1831, said: *“So protestiere ich gegen den Gebrauch einer unendlichen Grösse als einer vollendeten, welches in der Mathematik niemals erlaubt ist.”* (Meschkowski, p. 65). Kronecker: Dauben, pp. 66–70, and in particular p. 68.

236: [Almost all the world] Notable exceptions included Leibniz and Bolzano. Leibniz wrote in a letter (*Opera Omnia,* Studio Ludov. Dutens. ii part i, p. 243), *“Je suis tellement pour l’infini actuel, qu’au lieu d’admettre que la nature l’abhorre, comme on dit vulgairement, je tiens qu’elle l’affecte partout, pour mieux marquer la perfection de son auteur. Ainsi je crois q’il n’y a aucune partie de la matière, qui ne soit, je ne dis pas divisible, mais actuellement divisée; et par consequent la moindre particule doit être considérée comme un monde plein d’une infinité de créatures différentes.”* In his *Paradoxes of the Infinite* (published in1851), Bernhard Bolzano also asserted the existence of the actual infinite and pointed toward the idea of 1–1 correspondences between sets (Kline, p. 994). He anticipated Kronecker’s claim that God made the natural numbers but claimed that he had made infinite numbers as well. Zero, however, wasn’t a natural number for him (Novy, pp. 90–91).

237: [Gersau] Purkert (op. cit.), p. 43.

237: [Letter of Nov. 29, 1873] Noether, p. 12.

237: [Letter of Dec. 2, 1873] Noether, p.13.

237: [Cantor's later proof] The original proof dates from Dec. 7, 1873; the later version from 1890.

240: [Portrait of the middle-aged Cantor] From a portrait in the possession of Wilhelm Stahl (Dauben, p. 121).

242: [Hills looking like valleys] As the Red Queen said to Alice in the garden.

243: [Letter of Jan. 5, 1874] Noether, pp. 20–21.

244: [Cantor asks Dedekind if he too is having problems] Letter of May 18, 1874: Noether, p. 21.

244: [Letter of June 20, 1877] Noether, pp. 25–26.

244: [Postcard to Dedekind] Noether, p. 34.

244: [Churchill on Russia] Broadcast of Oct. 1, 1939.

245: [Exchange about Dedekind's objection] Noether, pp. 27–28.

245: [Einstein and confirmatory observations] This anecdote appears in Albrecht Folsing's *Albert EInstein, a Biography* (Viking, 1997), p. 439. Compare this remark about Crick and Watson, from Jim Holt's review of Brenda Maddox's *Rosalind Franklin: The Dark Lady of D.N.A.* (*The New Yorker*, Oct. 28, 2002, p. 106): "Watson and Crick's method was the opposite of Rosalind's: trust no datum until it had been confirmed by theory. They were determined to solve the structure of DNA with as few empirical assumptions as possible." The elevation of deduction over observation has a venerable history. In Galileo's *Dialogues,* Simplicio is made to ask whether an experiment had been made. "No," Galileo replies, "and I do not need it, as without any experience I can confirm that it is so, because it cannot be otherwise (Kline, p. 331, who adds: "Newton... too says that he used experiments to make his results physically intelligible and to convince the common people.")

245: [Objection of Du Bois-Reymond] Kline, p. 998.

246: ["Mathematics is freedom"] Cantor, *Collected Works,* p. 182.

248: [Jia Xian's triangle, and al-Karaji's] On Jia Xian's (mid-eleventh c.), see Katz, p. 202; on al-Karaji's (d. 1019), Katz, p. 258.

249: [Piffle before the wind] The words are from Daisy Ashford's immortal novel, *The Young Visiters.*

252: [Passage from Cantor] Dauben, p. 98 (quoting from the *Grundlagen*).

254: [Cantor's dozen year digression] In 1885 Cantor began to suspect that a more fruitful approach to the continuum problem might lie through a study of order (Dauben, p. 150). He published the results of this approach in his last major work, the *Beiträge*. Part I, on simply-ordered sets, appeared in 1895; Part II, on well-ordered sets, in 1897. See Dauben, chs. 8 and 9.

254: [Cantor's reasons for choosing aleph] For other possible reasons, see Dauben, p. 179.

254: [Cantor's Jewish ancestry] Because the Nazis stigmatized set theory as "Jewish mathematics"(see Purkert, op. cit., pp. 13–16), much work had been done in the thirties to prove that Cantor's lineage wasn't Jewish at all. This work may well have been aided by attempts of the family itself to disguise its origins. Here, however, is a sentence from a letter written by Cantor's brother Ludwig to his mother in 1869: (nr. 36 in *Cod. Ms. Georg Cantor* [Niedersächsische Staats-und Universitätsbibliothek, Göttingen])*: "Mögen wir zehnmal von Juden abstammen und ich im Prinzip noch so sehr für Gleichberechtigung der Hebräer sein, im socialen Leben sind mir Christen lieber."* That is: "Even were we ten times descended from Jews..."—an odd expression, by the way, since he doesn't say "ten times *more* (than we are) descended..."—and of course such descent tends to come in powers of 2. The Cantors as a family seem to have had a special relation to counting. Cantor's biographer Joseph Dauben emphatically says (p. 315): "In fact, Cantor was not Jewish." He adds in a footnote (p. 315, n. 4): "...the matter is not as clear as it might be, considering a reference Cantor once made to his *Israelitische* grandparents in a letter to P. Tannery on January 6, 1896."

255: [For at the gates...] You will not find this improvement on Swift in De Morgan.

255: [The Fisherman and his Wife] The similarity of Cantor's fate to that of Virginia Woolf's Mr. Ramsey, in *To the Lighthouse,* makes this Grimm tale especially apposite.

255: [Poincaré on the pathology of set theory] Dauben, p. 1: "Poincaré thought set theory and Cantor's transfinite numbers represented a grave mathematical malady, a perverse pathological illness that would one day be cured." Ibid. p. 268: "Poincaré (1908) said that most of the ideas of Cantorian set theory should be banished from mathematics once and for all." See too Kline, p. 1003.

255: [Kronecker's attacks on Cantor] Dauben, p.134, p.1, on Cantor as a hum-bug and, respectively, a charlatan.

255: [Cantor's complaints about poverty and recrimination] Dauben, p. 162.

255: [Cantor's sense of persecution] Dauben, p. 286.

255: [First breakdown] Dauben, p. 135.

255: [When two sets have the same cardinality] The theorem that two sets A and B have the same cardinality if and only if each can be put into 1–1 correspondence with a subset of the other, was proven by Schroeder and Bernstein in 1898.

255: [Schopenhauer's consoling himself] "If at times I have felt unhappy, that has been due, after all, only to a blunder, to a personal confusion; I have mistaken myself for someone else and complained of his woes: for instance, a *Privatdozent* who has not obtained his professorship and who gets no students; or for one maligned by a certain Philistine or gossiped about by a certain scandal-monger; or for the defendant in a law-suit for assault; or for a lover disdained by his precious maiden; or for a patient kept at home by his illness; or for such other persons afflicted with such miseries. But I myself have been none of all these; that was all alien fabric of which, let me say, my coat was made, which I wore for a while and then discarded for another. Who am I, then? The author of *The World as Will and Idea,* who has given the solution of the great problem of existence, a solution which perhaps displaces all previous ones, and which at any rate will keep busy the thinkers of ages to come. I am that man, and what can trouble him during the years that he still has to breathe?" From *Schopenhauers Gespräche und Selbstgespräche* (ed. E. Griesebach, 1902, p. 133 f.).

256: [Axiom of Choice] Zermelo stated the Axiom of Choice on Sept. 14, 1904 (Dauben, pp. 250–51).

257: [König's announcement] Dauben, pp. 248–50.

257: [Cantor at breakfast] Dauben, pp. 249–50.

257: [Passage from Sterne] *Tristram Shandy,* iv, ch. x.

257: [Passage from Bacon] Quoted in Dauben, p. 238.

257: [Cantor's letter to a friend] This was his letter to Mittag-Leffler. Dauben, p. 351, note 85.

258: [Hilbert] "On the Infinite", *Math. Ann.* 95 (1926), pp. 161–90.

258: [Was Cantor a formalist] Hallett, p. 19: Cantor "...had no idea of a formally presented theory."

258: [Cantor on real ideas in the divine intellect] Hallett, p. 21, quoting from an 1895 letter of Cantor's to Fr. Ignatius Jeiler. Dauben (p. 238) puts it well: "Cantor regarded the reality of the possible as guaranteed by its consistency..."

258: [Corporeal objects in the world] See Hallett, p. 23, on Cantor's ideas about "aether monads" and (hence) "monads of matter".

258: [Cantor and the apeiron] Dauben, p. 170.

258: [Burali-Forti] Because he was turned down as a doctoral candidate, he spent his life teaching in a military academy. The reason he was turned down was that he was a great defender of using those vectors we saw Euler, Wessel, and Argand invent; but vector methods were frowned on at the time. His vector carried him off into fierce independence from the conventional bases of Italian life, so that he even asked that he not be given a religious funeral. See the St. Andrews website: Burali-Forti.

259: [Nothing now more uncertain than mathematics] J. Thomae (Dauben, p. 242).

259: [Cantor sees paradox as beneficial] Dauben, pp. 242–43.

259: [*Battle of Maldon*] the original (lines 296–97) is:

Hige sceal the heardra, heorte the cenre,
mod sceal the mare, the ure maegen lytlath.

259: [Cantor's simultaneous humility and arrogance] Meschkowski, p. 165.

259: [Consistent and inconsistent sets] Cantor called a multiplicity inconsistent if "the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of [it] as a unity..." But if "the totality of elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into *'one* thing', I call it a *consistent multiplicity* or a 'set'." Cantor, letter to Dedekind of July 28, 1899, in Jean van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931* (Harvard University Press, 1971), p. 114.

259: [The absolutely infinite] See Hallett, pp. 42–45 on the meanings of "the absolutely infinite" for Cantor.

260: [Names of large cardinals] You will find many of these imaginative names in modern books on set theory (or as it is also sometimes called, Combinatorics), such as Thomas Jech's *Set Theory* (Academic Press, 1978).

260: [On the devising of these cardinals as a game] See Kenneth Kunen, "Combinatorics", in Jon Barwise, *Handbook of Mathematical Logic* (North Holland, 1983), p. 396: in this game "my goal is to try to completely demolish your ego by transcending your number via some completely new principle."

260: [Large cardinals enabling proofs of finite events] So, for example, Stephen Simpson ("Unprovable Theorems and Fast-Growing Functions", Penn State University, Department of Mathematics Research Report, 1985) discusses three theorems, each of which "is simple and elegant and refers only to finite structures. Each of these three theorems has a simple and elegant proof... each of the proofs uses an infinite set at some crucial point. Moreover, deep logical investigations have shown that the infinite sets are indispensable. *Any* proof of one of these finite combinatorial theorems *must* involve a detour through the infinite.” See too Hallett, pp. 101–3.

260: [Cantor’s proof and Zermelo’s critique of it] It was the improbability of such successive arbitrary choices that led Zermelo to formulate his Axiom of Choice, where the arbitrary choices are made all at once—as if that were any more palatable. See Hallett, p. 170.

260: [Russell’s paradoxes] Dauben, p. 262.

261: [Cantor on set as abyss] From Oskar Becker’s *Grundlagen der Mathematik in geschichtliche Entwicklung* (Verlag Karl Alber, Freiburg/München, 1954), p. 316: *“Zum Schluss sei noch eine hübsche Anecdote mitgeteilt, die nach F. Bernsteins Zeugnis Emmy Noether berichtet: sie kennzeichnet in sehr anschaulicher Weise die verschiedene gefühlmässige Haltung Dedekinds und Cantors zu der Vorstellung einer aktual unendlichen Menge.*

“F. Bernstein übermittelt noch die folgenden Bemerkungen, ... ‘Von besonderen Interesse dürfte folgende Episode sein: Dedekind äusserte, hinsichtlich des Begriffes der Menge: er stelle sich eine Menge vor wie einen geschlossenen Sack, der ganz bestimmte Dinge enthalte, die man aber nicht sehe, und von denen man nichts wisse, ausser dass sie vorhanden und bestimmt seien. Einige Zeit später gab Cantor seine Vorstellung eine Menge zu erkennen: Er richtete seine kolossale Figur auf,

beschrieb mit erhebenem Arm eine grossartige Geste und sagte mit einem ins Unbestimmte gerichteten Blick: "Eine Menge stelle ich mir vor wie einen Abgrund."""

261: [Quotation from Dauben] Dauben, p. 266.

261: [Cantor and the violin] Dauben, p. 283.

261: [Cantor's string quartet] Dauben, p. 278.

261: ["Do you still love me?"] Dauben, p. 288.

261: [Rosicrucians, Theosophy, Free Masonry] Dauben, p. 281.

261: [Shakespeare's plays written by Bacon] Dauben, pp. 281–83, 286.

261: [Discoveries concerning the first King of England] Dauben, p. 282, quoting a letter of Cantor's.

261: [Portrait of the old Cantor] From a photograph in the possession of Helga Schneider and Sigrid Lange (Dauben, p. 273).

262: [Ex Oriente Lux] Dauben, p. 289.

262: ["The time will come..."] I Corinthians (Dauben, p. 239).

262: [Passage from Playfair] John Playfair, "Life of Dr. Hutton", in *Transactions of the Royal Society of Edinburgh*, v, part iii (1805), p. 73.

262: [A new understanding...] Consider this passage from Stillwell, pp. 328–29:

"Gödel's theorem shows that something is missing in the purely formal view of mathematics... Despite this, the official view still seems to be that mathematics consists in the formal deduction of theorems from fixed axioms. As early as 1941 [the logician Emil] Post protested against this view:

> '...It has seemed to us to be inevitable that [Gödel's Theorem] will result in a reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth.'...

"I believe that what Post was saying was this. Before Gödel... it was expected that all of number theory, for example, could be recovered by formal deduction from [the Peano Axioms], that is, by *forgetting that [these] axioms had any meaning.* Gödel showed that this was not so... But it is precisely by knowing the *meaning* of [these] axioms that one knows they are consistent: contradictory statements cannot hold in the actual structure of $\mathbb{N}$ with + and • . Thus it is the ability to see meaning in [the Peano Axioms] that enables us to see the truth of [their consistency] and hence to transcend the power of formal proof."

262: [Blake on the doors of perception] From *The Marriage of Heaven and Hell*, "A Memorable Fancy". The passage is: "If the doors of perception were cleansed every thing would appear to man as it is, infinite."

APPENDIX

To Chapter Two

264: [Many models] It is intriguing to find how blithely mathematical logicians recognize that the slew of models they come up with are counter-intuitive and the 'model' of the numbers as we intuit them is of course The Real Thing. So Paul Cohen, in *Set Theory and the Continuum Hypothesis* (Benjamin, 1966) says of a certain model: "Since the construction of the model was rather indirect, we do not expect that the objects of the model will be identifiable with 'real' sets... in the case of set theory, or number theory, or the real number system, we have in mind one particular model, and we are primarily interested in it..." How we 'have it in mind' is left unexplored.

The chain of mental events seems to be this. We grasp numbers and their doings through intuition and practice, then realize that our grasp isn't strong enough, so replace it by the tighter hold that formalizing gives. This approach then yields one model (or instantiation) after another of that form, with the result that what it was meant to clarify is rethought as *no more than* another model of the now seemingly prior or deeper or *generative* form. So Schumann sensed his 'subjective self' to precede and be more authentic than the flurry of likenesses launched on the world by the split between form and shadow; yet at the same time grew so engrossed in the play of appearances that *all* these selves came to seem just—on a par.

265: [Nala and Damayantī] This story, the *Anabasis* of Sanskrit students, is in the third book of the *Mahābhārata*.

266: [Gödel, doppelgängers and the scaffolding] The difficulty mathematics finds itself in is this. Its axioms are expressed in terms of sets and their behavior. If the formal language containing these expressions is made up of logical connectives (such as “and”), symbols for members of sets (“x”, “y”) and quantifiers (“there exist”, “for all”) that refer to these members, the language is called *first-order logic.* If there are also symbols for sets and the quantifiers are allowed to refer as well to these, the language is called *second-order logic.* The distinction seems as trivial as a trip-wire.

Gödel showed that any formal system expressed in second-order logic must be incomplete: it will contain true statements which cannot be proven true within the system. This unsatisfactory situation threw the burden back on first-order logic. A Norwegian mathematician named Thoralf Skolem showed that set theory can be formulated in first-order logic. He and others also showed, however, that neither the system of real nor of natural numbers can be uniquely characterized in first-order logic: each will now have endlessly different models, so that we can no longer speak of “the” reals or “the” naturals. Damayantī must marry them all.

For a superb exposition of these matters, see Gregory H. Moore’s “A House Divided Against Itself: the Emergence of First-Order Logic as the Basis for Mathematics”, in Phillips, pp. 98–136.

266: [Hilbert’s voice] Our thanks to John Stillwell for calling our attention to this website.

To Chapter Three

269: [Euler named e in 1728] Kline, p. 258.

To Chapter Four

270: [Euler's proof] You will find it in (for example) Goldman, pp. 36–37.

270: [Clarkson's proof] From James A. Clarkson, "On the Series of Prime Reciprocals", in *Proceedings of the American Mathematical Society* 17: 541: MR32 (1966), #5573.

To Chapter Five

278: [Tenth point on nine-point circle] A proof by W. Weston Meyer is in *The American Mathematical Monthly* 108 no. 6, June–July 2001, p. 569. There are a few more significant points on this circle. Draw any line k through the orthocenter H of ∆ABC, and drop perpendiculars to it from A, B, and C, meeting k at L, M, and N. If you now draw a perpendicular from L, M, and N to the sides (perhaps extended) BC, AC, and AB respectively, these three perpendiculars will be concurrent—at a new point (called the orthopole) on the nine-point circle! For a proof, see Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry* (New Mathematical Library #37, 1995), p. 127. Since this is true for any of the infinitely many lines through H, we have just added infinitely many more points to the nine-point circle. There are more, as David Wells remarks (*The Penguin Dictionary of Curious and Interesting Geometry* [Penguin, 1991], p. 76). Here are some of them. The nine-point circle also has points of tangency with each of ∆ABC's three exocircles, and with its incircle. Since this nine-point circle is also the nine-point circle of triangles AHB, BHC, and CHA, it has points of tangency with each of these three triangles' in- and exo-circles—a total of 25 new points, therefore. And...

280: [Hofmann] Coxeter (p. 21) dates his proof to 1929. There are also proofs by Cavalieri (1647), Simpson (1710–1761), Crelle (1816), Heinen (1834) and Steiner (1842). See Kline, pp. 837, 839 and the St. Andrews website: Torricelli.

282: [Pons asinorum] Heath (Euclid, i, p. 415) gives us a little Sunday morning's puzzle in this earliest occurrence of the name, from an epigram of 1780 (recorded in Murray's English Dictionary):

> If this be rightly called the bridge of asses,
> He's not the fool that sticks but he that passes.

Why? Was its author a proto-intuitionist, recognizing that the more intricate the proof, the more seductive would formalism appear? Was his objection to the way 'of' is used, implying that those who cross it are asses? Was he one of those beef and beer Englishmen who knows that damned brains only get in the way of action, and that *purus mathematicus purus asinus* (as J. J. Sylvester wryly recalled in his Inaugural Address to the British Association in 1869)? Or was it just written reeling? Heath soberly comments that its writer's view is not too clear.

287: [Bhāskara and the quadratic formula] Brahmagupta (b. 598) gave the quadratic formula in just about the form we know it, but only mentions one solution. Bhāskara (1114–1185) deals with two, although he is hesitant about negative solutions to real problems, and gives no examples of quadratics with two negative or any irrational roots. (Katz, pp. 226–27).

288: [Hermes and his Diarium] We are immensely grateful to Paddy Patterson, at the Mathematische Institut in Göttingen, for his help in tracking down some of the stories about this curious affair. See too Hermann Tietze's *Gelöste und Ungelöste Mathematische Probleme aus Alter und Neuer Zeit* (1946), ii, p. 15 (translated as *Famous Problems of Mathematics* by B. K. Hofstadter and H. Komm in 1965).

To Chapter Seven

295: [Biographical details of Cardano] Kline, pp. 221–22; Stillwell, pp. 61–62; Katz, p. 330.

295: [Portrait of Cardano] From an unattributed engraving in Stillwell, p. 61.

295:[Passage from Cardano] Fauvel and Gray, p. 263.

302: [Wellington] *Creevey Papers,* x, p. 236.

304: [Pappus's proof] Kline, pp. 127–28.

304: [Möbius's proof] Stillwell, p. 85.

307: [Importing coordinates to the projective plane] In 1847 Karl Georg Christian von Staudt devised the needed analogues of length on the projective plane through his “Algebra of Throws”. See Kline, pp. 850–51. Angle-measure was imported to the projective plane by Edmond Laguerre in 1853 (Kline, pp. 906–7). Julius Plücker introduced the idea of homogeneous coordinates in 1831.

314: [Nietzsche] This is from *Thus Spake Zarathustra*, First Part, section 7: “On Reading and Writing”.