THE FOUR NUMBERS GAME, WEEK 2

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We return to the question of whether we can go backwards.

After some experimentation, we realized that we could go backwards if one of the entries was the sum of the other three entries. If $a = b + c + d$, then we can write $(0, b, b + c, b + c + d)$, noting that the last entry is just a, and therefore taking a step would give us (a, b, c, d).

So given a tuple (a, b, c, d), can we find four numbers that give us an equivalent tuple?

Suppose that $a = b + c + d + k$, where k is a positive whole number. Then if we add 1 to all the numbers to get an equivalent tuple $(e, f, g, h) = (a + 1, b + 1, c + 1, d + 1)$, we get that $e = f + g + h + (k - 2)$.

Thus we can just add constants until k vanishes. Hence given a tuple (a, b, c, d), we can go backwards from an equivalent game, and therefore make a game of any length.

Now we took a short look at what happens if we use real numbers instead of just rational numbers. After a brief excursion to define e as the limit of $(1+\frac{1}{n})^n$ as n goes to ∞, we asked whether $(e, \pi, 1, 0)$ terminates. Since multiplying it by various numbers didn't yield a tuple made completely of whole numbers, we decided to put this on hold for a moment.

Having dealt with the Four Numbers game for a while, we switched to the Five Numbers game. First we asked if the Five Numbers Game always terminated.

We looked at $(5, 4, 3, 2, 1)$. This yielded

$$(5, 4, 3, 2, 1)$$
$$(1, 1, 1, 1, 4)$$
$$(0, 0, 0, 0, 3)$$
$$(0, 0, 0, 3, 3)$$
$$(0, 0, 3, 0, 3)$$
$$(0, 3, 3, 3, 3)$$
$$(3, 0, 0, 0, 3)$$

We noted that this last step looks exactly like three steps before it, but shifted. Therefore we know that this particular game will never terminate, but will cycle instead. Therefore not all games will terminate in the Five Numbers game.

What about the Three Numbers Game? We looked at $(3, 2, 1)$, which gave us:

$$(3, 2, 1)$$
$$(1, 1, 2)$$

$$(0, 1, 1)$$
$$(1, 0, 1)$$

Thus we get that there are cycles in the Three Numbers Game as well.

We conjectured that for all odd n, the n Numbers Game will have such a cycle.

What about the even numbers? We know that the Four Numbers Game terminates, and we can show that any given Two Numbers Game terminates. So does the n Numbers Game terminate if n is even?

We looked at $(8, 6, 5, 4, 3, 2)$, which yielded

$$(8, 6, 5, 4, 3, 2)$$
$$(2, 1, 1, 1, 1, 6)$$
$$(1, 0, 0, 0, 5, 4)$$
$$(1, 0, 0, 5, 1, 3)$$
$$(1, 0, 5, 4, 2, 2)$$
$$(1, 5, 1, 2, 0, 1)$$
$$(4, 4, 1, 2, 1, 0)$$
$$(0, 3, 1, 1, 1, 3)$$
$$(3, 2, 0, 0, 2, 3)$$
$$(1, 2, 0, 2, 1, 0)$$
$$(1, 2, 2, 1, 1, 1)$$
$$(1, 0, 1, 0, 0, 0)$$
$$(1, 1, 1, 0, 0, 1)$$
$$(0, 0, 1, 0, 1, 0)$$

We note that the last step and the one right before it are the same, only shifted. So we still get cycles in the Six Numbers Game.

We noted that if we took alternate corners to make a triangle, one of the triangles became $(0, 1, 1)$, which is the cycle in the Three Numbers Game.

So we conjectured that if n is even but divisible by an odd number greater than 1, we could form a cyclic n game by taking an odd factor m and weaving copies of cyclic m games together.

So what if n is a power of 2? Can we describe an Eight Numbers Game as two Four Numbers Game? The answer at the end of the class seemed to be yes, but I'm not completely convinced.