# MAKING THE REALS FROM THE RATIONALS, WEEK 5-8

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After finding out that the embedding method didn't work, we had two options: we could either try to salvage the method, or turn to something else.

We decided to look at something that had come up previously: Adding modulo 2 instead of taking the absolute difference.

We looked at a few examples, and found that if we take the 4 Numbers game and wrote the initial values as $(a, b, c, d)$, the following several entries would be:

$$(a + b, b + c, c + d, d + a)$$

$$(a + 2b + c, b + 2c + d, c + 2d + a)$$

$$(a + 3b + 3c + d, b + 3c + 3d + a, c + 3d + 3a + b, d + 3a + 3b + c)$$

$$(2a + 4b + 6c + 4d, 2b + 4c + 6d + 4a, 2c + 4d + 6a + 4b, 2d + 4a + 6b + 4c)$$

Note that in the fourth row all of the coefficients are even, and hence mod 2 everything would disappear.

By previous work, we could then show that we would eventually get all even numbers in the subtraction version, and hence if we could show that the addition mod 2 game always terminated, the subtraction game would get to a point of being able to divide by two, and thus the game would be equivalent to a game with smaller numbers, until the subtraction game also terminated.

So how do we show that for a general $n$-game, we eventually get all even entries?

We discovered that if instead of writing

$$(2a + 4b + 6c + 4d, 2b + 4c + 6d + 4a, 2c + 4d + 6a + 4b, 2d + 4a + 6b + 4c)$$

we wrote the last line as

$$(a + 4b + 6c + 4d + a, b + 4c + 6d + 4a + b, c + 4d + 6a + 4b + c, d + 4a + 6b + 4c + d)$$

we get that the coefficients of each step spell out Pascal's triangle:

1

11

121

1331

14641

Why does it write out Pascal's triangle?

We consider two adjacent entries:

$$(1 * a + n_1 * b + n_2 * c + n_3 * d, 1 * b + n_1 * c + n_2 * d + n_3 * a)$$

Hence we get

$$((1+n_3)*a, (1+n_1)*b, (n_1+n_2)*c, (n_2+n_3)*d, etc)$$

We see that we get adjacent coefficients in the previous step adding to make coefficients in the next step, with overlap between the last coefficient and the first.

Ignoring the overlap, we get two numbers next to each other adding to make the next number, just as in Pascal's triangle.

So if we can show that the $n$th row from Pascal's triangle is all even except for the ends, we know that an $n$-game with addition modulo 2 will terminate, and hence an $n$ game with subtraction will terminate.

Specifically, we want to show it for $n = 2^k$ for some whole number $k$.

What about if $n$ is not a power of 2?

Well, let's look at what happens with a five-number game:

$$(a, b, c, d, e)$$

$$(a+b, b+c, c+d, d+e)$$

$$(a+2b+c, b+2c+d, c+2d+e, d+2e+a, e+2a+b)$$

$$(a+3b+3c+d, b+3c+3d+a, c+3d+3a+b, d+3a+3b+c, e+3a+3b+c)$$

$$(a+4b+6c+4d+e, b+4c+6d+4e+a, c+4d+6e+4a+b, d+4e+6a+4b+c, e+4a+6b+4c+d)$$

$$(a+5b+10c+10d+5e+a...)$$

Note that neither the fourth nor the fifth row are all even, since the overlap has been moved to the fifth row and the middle entries of the fifth row are not all even.

So what about the $2^k$ games? What can we say about the $2^k$th row of Pascal's triangle?

Well, it turns out that the $i$th entry (if we count the first 1 in each row as the 0th entry) of the $j$th row of Pascal's triangle is equal to $\binom{j}{i}$, otherwise known as the binomial coefficient or $j - choose - i$. It is called $j - choose - i$ because it is the number of ways to choose $i$ objects out of $j$ objects, assuming that order doesn't matter. It is called the binomial coefficient because if you expanded $(a+b)^j$ and considered the coefficient of the $a^i$ term, you'd get $\binom{j}{i}$.

Hence we want to show that $\binom{j}{i}$ is even for $j = 2^k$ and $i$ not equal to 0 or $2^k$.

We have the helpful formula

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}$$

We can rewrite this as

$$\binom{j}{i} = \frac{j(j-1)(j-2)(j-3)...(j-i+1)}{i(i-1)(i-2)...3*2*1}$$

Since $j = 2^k$, we know that for $m < i$, we have that $m$ and $j - m$ have the same number of 2s. Hence we can ignore all factors in the numerator and the denominator except for $j$ and $i$. Since $j = 2^k$, $i$ definitely has fewer factors of 2 than $j$ does unless $i = 0$ or $j$.

Therefore $\frac{j(j-1)(j-2)(j-3)...(j-i+1)}{i(i-1)(i-2)...3*2*1}$ is even, and hence for $j = 2^k$, $\binom{j}{i}$ is even for $i$ not equal to 0 or $j$.

Thus a $2^k$ game with addition becomes all even, and hence a $2^k$ game mod 2 terminates, and hence any $2^k$ game with subtraction terminates.

But we still have one more thing to make sure of: is $\frac{j!}{i!(j-i)!}$ actually an integer? It is the number of ways to choose $i$ things out of $j$ things, but that is not proof that $\frac{j!}{i!(j-i)!}$ is in fact an integer!

Writing it out as

$$\frac{j(j-1)(j-2)(j-3)...(j-i+1)}{i(i-1)(i-2)...3*2*1}$$

we find that we need to show that $j(j-1)(j-2)(j-3)...(j-i+1)$ is divisible the product of by every number less than or equal to $i$.

There are $i$ factors in $j(j-1)(j-2)(j-3)...(j-i+1)$, and hence there must be at least one of them divisible by $i$, and one of them divisible by $i-1$, and so on. So $j(j-1)(j-2)(j-3)...(j-i+1)$ is divisible by all numbers from $i$ downward. But we still need to show that it's divisible by their product.

We don't have to worry about it not being divisible by $i(i-1)$ if it's divisible by $i$ and $i-1$, since $i$ and $i-1$ are relatively prime, i.e. they have no common divisors except for 1.

In fact, the first number we have to worry about is $i/2$, assuming $i$ is even. So we check to see how many factors of $i/2$ are in $j(j-1)(j-2)(j-3)...(j-i+1)$.

It turns out that there are at least 2 in $j(j-1)(j-2)(j-3)...(j-i+1)$, and hence even though we lose one to $i$, we still have another one. Similarly, for $i/3$, we lose one to $i$, and another to $2i/3$, but we still have a third factor of $i/3$ somewhere.

Hence we get that $j(j-1)(j-2)(j-3)...(j-i+1)$ is in fact divisible by $i!$, and hence that

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}$$

is in fact an integer.

Hence we're done.

On a final, somewhat irrelevant note, we asked what was $\binom{1\frac{1}{4}}{\frac{3}{4}}$. I proposed that, at least for $\binom{x}{n}$ where $n$ is an integer, we generalize from the formula above and write

$$\binom{x}{n} = \frac{x(x-1)(x-2)...(x-n+1)}{n!}$$

and that this works even if $x$ is not a whole number. So, for example,

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} = \frac{1}{16}$$

We noted the oddity that

$$\binom{\frac{1}{2}}{2}$$

was negative, the meaning of which is still nebulous.