MAKING THE REALS FROM THE RATIONALS, WEEK 2

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We started out by asking how to do arithmetic on our sequences. We decided that both addition and multiplication would be pointwise, i.e. that

$$(0,0,0,0,...) + (1,1,1,1,...) = (1,1,1,1,...)$$

and

$$(0,0,0,0,...) \cdot (1,1,1,1,...) = (0,0,0,0,...)$$

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I then asked what would happen if we tried to use $(1,0,1,0,...)$ as $\frac{1}{2}$.

Adding $(1,0,1,0,...)$ and $(0,1,0,1,...)$ yields $(1,1,1,1,...)$, as we'd expect, but multiplying them gave $(0,0,0,0,...) \neq \frac{1}{4}$. Thus we showed that $(1,0,1,0,...)$ and $(0,1,0,1,...)$ are not valid representations of $\frac{1}{2}$.

We then decided to say that arbitrarily close meant that if a_n is within some interval $[a-\epsilon, a+\epsilon]$, then a_{n+1} should be between a_n and a, i.e. within $[a - |a - a_n|, a + |a - a_n|]$.

So then I asked about the sequence $(0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, ...)$. We decided that this sequence did in fact go to 0, but it violated our definition of arbitrarily close. So we had to backtrack a bit.

Then I asked if we could try making a definition that doesn't mention what the sequence goes to. We decided that the differences between entries ought to get smaller. But $(1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, ...)$ doesn't go to a real number, despite the differences getting smaller.

We also looked at the decimal expansion of the square-root of 2, $(1, 1.4, 1.41, 1.414, 1.4142, ...)$. First we had to decide whether we could even consider this sequence, since we couldn't predict the next number in the decimal sequence.

We compared this to a modification of the $(0, \frac{1}{2}, 0, \frac{1}{3}, 0, ...)$ sequence, where instead of alternating between positive fractions and 0s, the fractions would be separated by a random number of 0s.

We eventually decided that you could in fact use a random sequence as long as it obeyed the arbitrarily close rule.