# MATH CIRCLE - SET THEORY WEEKS 1 & 2

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## 1. Introduction: What are sets?

We started off by considering our favorite sets. Various examples were mentioned, such as the set of all real numbers (which contains $\pi, e$ and 2, for example) and the set of all people from Mars in our class. (How many people are in that set?)

But what *is* a set? Basically, we decided, it's a collection of objects which we can write down like

$$\{A, B, C, D\}$$

or perhaps

$$\{\text{all } x \text{ such that } x \text{ is an integer (whole number) greater than } 3\}.$$

We could also write the second set as

$$\{4, 5, 6, 7, \ldots\}.$$

The second example demonstrates that sometimes sets are defined by **rules**. More on this below.

Now what sort of things do we put in sets? So far, absolutely anything.

## 2. Being inside: "in" versus "within"

Here are some objects we considered.

$$A = \text{ the set of all animals,}$$

$$B = \text{ the set of all bats,}$$

$$b = \text{ a friendly bat named Billy.}$$

We tried to figure out the various relationships between $A, B$, and $b$. By drawing a picture with $A$ as a big circle, $B$ as a smaller circle inside it, and $b$ as a single point inside $B$, we observed that there were two sorts of "in" for set. On the one hand, $b$ is *in* $B$ and $b$ is *in* $A$. (Since it is a point inside either circle.) This reflect the fact that Billy the bat is both a bat and an animal. On the other hand, $B$ is also *in* $A$ since it sits inside it as a smaller circle.

Are these two notions of "in" the same? Not quite, it was decided. For example, $B$ is not *in* $A$ in the sense of being a point inside a circle. (I.e. in the way that $b$ is in $B$.) This is because $B$ is a set, the set of all bats, and a set is not an animal; so it doesn't belong in the set of all animals. But $B$ does "sit inside" $A$; it lies "within" $A$. So therefore the idea of "in" must be different from the idea of being "within". However, we could also consider

$$C = \text{ the set of all } \textit{sets} \text{ of animals.}$$

What would an object in $C$ look like? Any set which contains only animals. For example

$$\{\text{a bat, a cow, a tiger, a human}\}$$

is in $C$. It is also true that $B$ is in $C$, since every bat is an animal, so the set of all bats is a set of animals. Note that we have

$$b \text{ is in } B \text{ which is itself in } C.$$

Various names were proposed for the concepts "in" and "within" as described above. I think "sits inside" was also proposed, as was the idea of say "in, as a point" for the relationship between $b$ and $B$, and "in, as a circle" for the relationship between $B$ and $A$. (Draw a picture of the circles $A$ and $B$ and the point $b$ to make sure you have this idea straight.) We ultimately settled on "**in**" and "**within**" because they are shorter to write down. However, in the second class, Linus proposed the term "**subset**" for the idea of being "within" (that is, inside "as a circle"). We say that $B$, for example, is a subset of $A$ because everything in $B$ is also in $A$. If you draw a picture, a set $S$ will be a subset of a set $T$ if it can be drawn as a smaller circle inside a bigger circle. We can decide which term we like best later. There was one more term which was introduced, but hasn't yet been used: Linus felt there should be a name for the relationship between $b$ and $C$ noted above, where $b$ is in $B$, which is in turn in $C$. Arbitrarily, I decided to call this "**linusizes**": we say that $b$ linusizes $C$ in this case. We will see later whether this concept is a fruitful one to think about.

## 3. When are two sets equal?

We then tackled a thorny question: when are two sets equal to each other? Now nobody objected to the idea that the sets

$$\{1, 2, 3\}$$

and

$$\{1, 2, 3\}$$

are equal to each other... after all, they're the same set! However, consider the sets

$$\{\text{ all } x \text{ such that } x \text{ is the smallest whole number greater than } 1\}$$

and

$$\{2\}.$$

Or even trickier, the sets

$$B = \text{ the set of all blue-haired people in our class}$$

and

$$G = \text{ the set of all green-haired people in our class.}$$

Are $B$ and $G$ equal to one another? Note that their are *no* blue- *or* green-haired people in our class: both $B$ and $G$ are empty!

There were two factions of people in our class, at least in the first week. Some, like Ana Maria, felt that two sets should be equal to one another if and only if they are **defined by the same rule**. Others, like Mathilde, felt that two sets should be equal to one another if and only if they **have the same things inside them**. For example, $B = G$ in the second sense, but not in the first sense. Note that if we think of sets as circles and the things inside them as points within the circles, then the second idea of equality corresponds to the circles enclosing *exactly* the same area. However, two sets which enclose the same area in this manner might not be defined by the same rule.

Which is the "right" notion of equality for sets? This was left unsettled even after the second class. We ultimately decided to preserve both ideas, since each one makes sense in its own way. But this means we have to be careful when we write down that two sets are equal. If we want to say the sets have the same defining rule, we write $S =_r T$ where the $r$ stands for "rule". If we want to say the sets have the same objects in them, we write $S =_a T$ where the $a$ stands for "area".

## 4. Can we make any set we want?

At the end of the first week, someone mentioned the example of the set

$$S = \{ \text{ all } x \text{ such that } x \text{ is not in any set}\}.$$

Is this a "legitimate" definition for a set? Naïvely at least, it seems to be, since we wrote it down just fine. Can't we write down any set we want? But, it was pointed out, there's something fishy about this example. Because if there is any object $x$ inside $S$, we suddenly have a problem, since $x$ is in $S$ in contradiction to the definition of $S$ as containing *all and only* those objects which are not in *any* set! Is this paradoxical?

Eventually I believe we decided that $S$ is okay, as long as $S$ is empty. After all, we only get a contradiction if we suppose that there is some object $x$ in $S$. As long as $S$ contains no objects, we're okay! So we just have a weird way of writing down the empty set, $\{\}$.

However, this started an interesting discussion. The first observation to make is that *sets might be able to contain themselves.* For example, suppose that there is a set $V$ which contains *all* sets. Since $V$ is itself a set, it had better be true that $V$ is in $V$! Another example:

$$T = \{\text{all } x \text{ such that } x \text{ contains at least one object.}\}$$

Note that the set $\{1\}$ contains at least one object, so we must have that $\{1\}$ is in $T$. On the other hand, now we know that $T$ contains at least one object, namely $\{1\}$. So $T$ must be in $T$!

So what if we define a set

$$R = \{\text{all } x \text{ such that } x \text{ is } \textit{not} \text{ in } x\}?$$

Question: **Is $R$ in $R$?**

Suppose that $R$ is *not* in $R$. Then by definition, $R$ is a set which does not contain itself. Since $R$ contains **all** those sets which do not contain themselves, it follows that $R$ is in $R$. But this contradicts our supposition! Since we made an assumption and got a contradiction, our original assumption must have been false. So $R$ *is* in $R$. But $R$ consists of *exactly* those sets which do *not* contain themselves. Therefore, since $R$ *does* contain itself, $R$ *cannot* be in $R$! But $R$ *is* in $R$! Another contradiction! What are we to do?

This paradox was first observed around 1902 by the famous British logician Betrand Russell, and it causes a crisis in mathematics. Starting in week 3 we will have to figure out our own way to respond to this crisis. Maybe, it wasn't so safe to let just *anything* we wanted be a set after all!

## 5. Self Reference

A final thought. An idea behind Russell's paradox is *self-reference*, which was also famously used by the mathematician Georg Cantor in the 19th century to prove that **you cannot count all the real numbers**. For our purposes, a real number is just a decimal number like $2.0000\ldots$ or $1.547865721243545756345091234567876543456786543\ldots$.

How did Cantor's proof work? Well, suppose we *could* count the real numbers. Then we would have a list like

$$\begin{array}{r} 1.343346546762123423457 8\ldots \\ 0.3423549879237498327432\ldots \\ 6.0000000000000000000000\ldots \\ 19.1767657235325456435 43\ldots \\ 0.0987654356782345674321\ldots \\ \vdots \end{array}$$

Now let's make a new number $x$ as follows. Start with 0. Now to get the *first* digit after the decimal point, take the *first* number on our list, look at its *first* digit after the decimal point, and change it to something different. For example, we could just add one to it, unless it is a 9 in which case we can make it a 0. Similarly, to get the *second* digit after the decimal point of $x$ we take the *second* number on our list, look at its *second* digit after the decimal point, and change it as above. Continuing in this manner we can get an infinite decimal expansion for $x$. In the example above we would have

$$x = 0.45187\ldots$$

Now, since we have a list of *all* the real numbers, $x$ must show up as some item on the list. For example, suppose it is the 436th number $x_{436}$. But it can't be! Look at its 436th digit: we got it by taking the 436th digit of $x_{436}$ and adding 1. So certainly $x \neq x_{436}$. Similarly, if $x = x_N$ for any $N$ then comparing the $N$th digit after the decimal points we see that in fact $x \neq x_N$. So $x$ can't be *anywhere* on our list! Therefore we must not be able to make a list of all the real numbers at all.

In the coming weeks, we'll see other funny things that happen when we deal with infinite sets of numbers.