MATH CIRCLE - SET THEORY WEEK 3

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Today we reviewed the crisis we ran into last week, where a set

$$R = \{\text{all } x \text{ such that } x \text{ is not in } x\}$$

turned out to be contradictory. (Because R can be neither *in* R nor *not in* R.)

Our response: we had better *forbid* sets to contain themselves! But how do we do this? We need to very carefully start our set theory over again, saying *exactly* which sets are and aren't allowed to exist. We thus need to carefully define a **dictionary** of the terms we use, such as **set, in, within, element, rule, null set,** etc.

We next turned to the thorny question of **does the empty set** $\{\}$ **exist**? Some said yes, and some said no. The key thing to keep in mind is the **distinction between nothing** (what is *in* the empty set) and *the set with nothing in it* (i.e. the empty set). Just because the empty set is empty, doesn't mean it's nothing! The metaphor we used was *an empty cardboard box in the garage.*

In building up our set theory from scratch, we decided that it was best, for now, to allow only sets built up from sets we already know exist. To see whether this is sufficient, it is necessary to try to define ordinary things we want to put into sets (like numbers) in terms of sets. We began trying to do this. The first question is, what should the number 0 be, as a set? One idea was: nothing! After all, zero is nothing, isn't it? But no, someone suggested that although in some sense zero means nothing, it is itself actually something – namely zero! A proposal which seemed promising was to make 0 the set $\{\}$. Joseph then proposed the following scheme for making new numbers:

$$\begin{gathered}
\text{“0”} = x_0 = \{\} \\
\text{“1”} = x_1 = \{\{\}\} \\
\text{“2”} = x_2 = \{\{\{\}\}\} \\
\vdots \\
\text{“}n\text{”} = x_n = \overbrace{\{\{\{\cdots\{}^{n+1}\}\cdots\}\}\} \\
x_n = \{x_{n-1}\}
\end{gathered}$$

The last is a sort of “recursive” or “inductive” definition of x_n in terms of x_{n-1}.

Next week we will have to see whether we can turn this scheme into a genuine arithmetic using only sets! Crucial will be the idea of what sorts of operations we can do with sets, which might correspond to addition, subtraction, etc. For example, we can join sets together:

$$\{A, B, C, D\} \text{ joined with } \{C, D, E, F\}$$

is the set

$$\{A, B, C, D, E\}$$

(note that C and D are already in the first set, so we don't need to add them again when we join it with the second set). How can we define operations like this formally, and carefully, using only the precise language of our set dictionary?