

MATH CIRCLE - SET THEORY WEEK 4

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We were working on two projects: (1) Define mathematics starting only with set theory! That is, use set theory as a *foundation* for ordinary math. We proceeded to try to do this by starting with ordinary counting numbers. After a little debate, we ultimately agreed that we must define the numbers using only sets of sets, not sets of numbers. What should 0 be? Eventually it was settled that it should be the null set $\{\}$, for lack of a better alternative, although there was some disagreement.

Side note: is there a *rule* defining the empty set? Yes! We realized that we can define the empty set as the set of all x such that $x \neq x!$ (Except that first we have to say what we mean by $=$. We already have two possible definitions for this, but in building up a *formal* set theory, we have to choose one.)

In trying to define the counting numbers, one observation that was made is that the operation of “adding one” is somehow fundamental: if we have 0 (and we do, as $\{\}$) then we can get all the numbers provided we know how to increment by one. We had one proposal for this already, namely for any number x we define the next number $x' = \{x\}$ to be the set containing only x . Think of x' as “ $x + 1$ ”, except we don’t yet know what $+$ is! What *should* $+$ be? Once again, the suggestion of using the *union* of two numbers as their sum was proposed. But remember we defined $0 = \{\}$, $1 = \{0\} = \{\{\}\}$ and $2 = \{1\} = \{\{\{\}\}\}$. Certainly we should have $1 + 1 = 2$. But if we use unions then

$$1 + 1 = \{\{\}\} \text{joined with } \{\{\}\} = \{\{\}\} = 1$$

since the set containing only $\{\}$ already contains all the elements of the set containing only $\{\}$. Note that we might also write this, to use a popular notation, as

$$1 + 1 = \{\{\}\} \cup \{\{\}\} = \{\{\}\} = 1.$$

This is the wrong answer! So addition is not yet settled... we’ll have to come with another way. And how about multiplication, subtraction, etc?

Side note: We found that certain constructions such as *ordered pairs* (x, y) would be useful for defining things like rational numbers, and points of the plane, assuming we can make the numbers. How can we define (x, y) for sets x and y ? After all, in the *set* $\{x, y\}$, unfortunately, *order doesn’t matter*!

Our second project was (2) to start a *formal*, carefully-defined set theory that would be free from the Russell Paradox contradiction (i.e. the set of all sets which do not contain themselves). As Joseph pointed out, the problem with this paradox was that we were too freehanded about what sorts of rules we allowed ourselves to use in defining sets. We would *like* to say that any rule defines a set, but this runs into problems. So how can we restrict the rules to keep ourselves safe from contradiction?

This is where we’ll pick up next time...