MATH CIRCLE - SET THEORY WEEKS 5-7

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It's been a few weeks since I put notes online. Sorry for the delay! Let me try to reconstruct some of what was covered in the class since Week 5.

The necessity of axioms

In our foundational project, we determined that it is necessary to come up with explicit rules for what sets are permitted to be sets! Otherwise, we might run into Russell-like paradoxes...

Can't we just make the rule, only non-paradoxical sets are allowed? This, we decided, was impractical, because it can be hard to decide if a putative set causes paradoxes. What if only a very difficult theorem, such as Fermat's Last Theorem ($x^n + y^n = z^n$ has no solutions (x, y, z) in the integers, if $n \geq 3$) could reveal the paradox? We would be left uncertain as to whether the sets we want to use are OK.

A better idea is to build up complicated sets we want from simple ones which we know are OK. Thus we need an axiom system that starts with some simple sets and lets us build biggers ones from them.

A formal language

What language do we write our axioms in? We already know that we have to use carefully-defined set-theoretic language, rather than plain English, to phrase our axioms. But in fact, we want something even stricter. We should phrase our axioms using a carefully-defined, and very small, alphabet of symbols. We then allow ourselves to manipulate these symbols only according to certain rules. The resulting formulas, made up of our symbols, will tell us which sets do and do not exist.

We decided on the following set of symbols.

- Variables $x, y, z, \ldots$
- 'For all' $\forall$
- 'There exists' $\exists$
- 'Not' $-$
- 'And' &
- 'Or' /
- 'Implies' $\Rightarrow$
- 'If and only if' $\Leftrightarrow$
- 'In' $\odot$
- 'Within/Subset' $\circledcirc$

- 'Equals' =. (*Note!* Since we're not yet comfortable making RULES, as this is where the paradoxes come from, we have to note that this = means "have the same elements." E.g. "The set of all bluehaired people in our class"="The set of all greenhaired people in our class".)

We maniputе formulas made up of these only according to logical rules. For example, if p and q and r stand for certain sentences made up of the symbols, then from $(p \Rightarrow q)\&(q \Rightarrow r)$ we can deduce the sentence $p \Rightarrow r$. What is the English translation of this rule? Why does it make sense?

This particular rule has a long history in logic, going back to the Middle Ages. It is known in Latin as *modus ponens*.

Reducing unnecessary symbols?

In fact, not all of the symbols are necessary. We discovered that it is possible to eliminate some from our language without loss of expressive power.

For example, we can define 'Subset' in terms of 'In' as follows. We define $a \circledcirc b$ to be the sentence

$$\forall x(x \odot a \Rightarrow x \odot b).$$

(Why does this say what we want it to?)

We can also eliminate some of the purely *logical* operations. For example, $p \Leftrightarrow q$ can be written $(p \Rightarrow q)\&(q \Rightarrow p)$.

For fun, try finding ways to express all of our logical symbols $\forall, \exists, \&, /, \Leftrightarrow, \Rightarrow$ in terms of only two: $\forall$ and a new symbol $|$ (called the 'Sheffer stroke' after a mathematician named Sheffer who discovered it). The symbol $|$ (which is sometimes called NAND for "not AND") is defined by $p|q$ means $-(p\&q)$. So the truth table is

p	q	$p \mid q$
0	0	1
0	1	1
1	0	1
1	1	0

I claim we can really reduce our alphabet to $|, \forall, \odot$! Can you see how?

But for clarity, we will keep the whole alphabet we came up with above. It's not logically necessary, but it helps make our language easier to understand.

A start on some axioms

One axiom to start with says that whenever we have a set x we can form the set $\{x\}$ containing only x as an element. How can we formulate this precisely?

In our language, it would be written

$$\forall x \exists y \forall z(z \odot y \Rightarrow z = x).$$

The set y whose existence is guaranteed by this axiom is the one we call $\{x\}$; its only element is x. Can you see how the formula says this?

We still don't have any sets at all! We need to start with *something*! A reasonable place to start is the empty set $\varnothing = \{\}$. An axiom guaranteeing its existence is

$$\exists \varnothing \forall x(x \in \varnothing \Rightarrow x \neq x).$$

This defines $\varnothing$ as containing only those sets which are not equal to themselves! Since there are no such sets, $\varnothing$ is empty, as we will attempt to prove in Week 8. I.e. we will try to prove the theorem

$$\forall x(-(x \odot \varnothing)).$$

Functions and powersets

To motivate the next sort of axiom, we take a moment to think about *subsets* and *functions*. Functions are a basic idea in set theory. We want to map each element of one set to an element of another set. How about functions $S \to \{0,1\}$ with values in a 2-element set? These turn out to be *the same thing as subsets*! For if $A \circledcirc S$ we define $f : S \to \{0,1\}$ by $f(s) = \begin{cases} 1, & s \odot A \\ 0, & -(s \odot A) \end{cases}$ Conversely, if $f : S \to \{0,1\}$ is a function we define $A \circledcirc S$ to be the set $\{x \odot S$ such that $f(x) = 1\}$. (We need to use a *rule* to define this set, but that's a topic for Week 8...)

Hence, if we want to talk about functions, as a special case we need to be able to talk about *subsets*. Therefore, it's worth having an axiom that says *the set of all subsets of a set is a set.*

$$\forall x \exists y \forall z(z \odot y \Leftrightarrow z \circledcirc x).$$

We'll discuss this further in Week 8.

That's about it. We'll pick up in Week 8 with trying to get our axiom system down to a science, then see what sort of things we can prove using it. (And maybe what sort of things we can't?)

Something to ponder: is it okay to have infinite sequences of sets

$$\cdots \odot x_n \odot x_{n-1} \odot \cdots \odot x_3 \odot x_2 \odot x_1?$$